Loop group method for CMC surfaces in $\mathbb{S}^3$
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These notes expand on a 2 hours mini-course given at the $\mathcal{H}$-workshop 2023 at IMAG in Granada. The goal was to give a self-contained introduction to the loop group method for CMC surfaces in the 3-sphere. It's a pleasure to thank José Miguel Manzano and Francisco Torralbo for the invitation.

In [1], Dorfmeister, Pedit and Wu gave a Weierstrass-type representation of harmonic maps into symmetric spaces. As an application, in the last section of the paper, they obtained a Weierstrass-type representation of CMC surfaces in euclidean space. This became known as the DPW method and gives a Weierstrass-type representation of CMC surfaces in various spaces, including $\mathbb{S}^3$, $\mathbb{H}^3$, also minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, AdS^3...

1 The DPW method in $\mathbb{S}^3$

1.1 Matrix model of $\mathbb{S}^3$

The point $x = (p, q) = (x_1 + ix_2, x_3 + ix_4) \in \mathbb{C}^2 \sim \mathbb{R}^4$ with the matrix

$$X = \begin{pmatrix} p & q \\ -\overline{q} & \overline{p} \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

We have $\det(X) = \|x\|^2$, so this identifies $\mathbb{S}^3$ with $SU(2)$.

The product $SU(2) \times SU(2)$ acts isometrically on $\mathbb{R}^4$ by $(U_1, U_2) \cdot X = U_1 X U_2^{-1}$. This gives the standard 2:1 covering map $SU(2) \times SU(2) \to SO(4)$ which is the universal cover of $SO(4)$.

The tangent space to $\mathbb{S}^3 \sim SU(2)$ at $e = (1, 0, 0, 0) \sim \mathrm{Id}$ is the 3-plane $x_1 = 0$, which is identified with the Lie algebra $\mathfrak{su}(2)$. For x, y in $T_e\mathbb{S}^3 \sim \mathfrak{su}(2)$, we have

$$\langle x, y \rangle = \tfrac{-1}{2}\mathrm{tr}(XY),$$

$$x \times y = \frac{1}{2}[X, Y]$$

where $x \times y$ is the cross-product and $[X, Y] = XY - YX$.

1.2 Loop groups

Let $\mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and $\mathbb{D} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. Let G be a matrix Lie group. We define the following loop groups (which are infinite dimensional Lie groups):

$$\Lambda G = \{u : \mathbb{S}^1 \xrightarrow{C^\infty} G\}.$$

$$\Lambda^+ G = \{u \in \Lambda G \mid u \text{ extends holomorphically to } \mathbb{D}\}.$$

The same notations will be used for the loop algebras associated to a Lie algebra. Let

$$\mathcal{T} = \left\{ \begin{pmatrix} \rho & \mu \\ 0 & \rho^{-1} \end{pmatrix} : \rho > 0, \mu \in \mathbb{C} \right\}.$$

Recall the standard QR decomposition:

$$SU(2) \times \mathcal{T} \xrightarrow{\sim} SL(2, C), \quad (Q, R) \mapsto QR$$

There is an analog of the QR decomposition for loop groups, which we only state in the case of $\Lambda SL(2, \mathbb{C})$. Define

$$\Lambda^+_{\mathbb{R}} SL(2, \mathbb{C}) = \{u \in \Lambda^+ SL(2, \mathbb{C}) \mid u(0) \in \mathcal{T}\}$$

Theorem 1 (Iwasawa decomposition)**.** *The product map*

$$\Lambda SU(2) \times \Lambda^+_{\mathbb{R}} SL(2, \mathbb{C}) \xrightarrow{\sim} \Lambda SL(2, \mathbb{C}), \quad (F, B) \mapsto FB$$

is a diffeomorphism. Any loop $\Phi \in \Lambda SL(2, \mathbb{C})$ can be decomposed in a unique way as $\Phi = FB$ with $F \in \Lambda SU(2)$, called the unitary part, and $B \in \Lambda^+_{\mathbb{R}} SL(2, \mathbb{C})$, called the positive part.

In particular, for constants loops, Iwasawa decomposition is just the standard QR decomposition.

Remark 1. *Many authors use "twisted" loops, following [1]. Here we are in the "untwisted" setup. There is in fact a variety of DPW methods which differ in details like this. This can be confusing for the beginner as this changes some formulas.*

1.3 DPW potential

Let Σ be a Riemann surface. A DPW potential ξ on Σ is a matrix-valued holomorphic 1-form on Σ, depending on the variables $z \in \Sigma$ and $\lambda \in \mathbb{C}$, such that $\lambda\xi \in \Lambda^+\mathfrak{sl}(2,\mathbb{C})$. In other words, ξ can be written in local coordinate z as

$$\xi(z,\lambda) = \sum_{k=-1}^{\infty} \lambda^k \xi_k(z) dz$$

with $\xi_k(z) \in \mathfrak{sl}(2,\mathbb{C})$ depending holomorphically on z.

1.4 DPW method, local version

Assume $\Sigma \subset \mathbb{C}$ is a simply connected domain. The input data for the DPW method is a DPW potential ξ on Σ and an initial condition $(z_0, \phi_0) \in \Sigma \times \Lambda SL(2,\mathbb{C})$.

Theorem 2. *Let $\Phi : \Sigma \to \Lambda SL(2,\mathbb{C})$ be the solution of the Cauchy problem*

$$\begin{cases} d\Phi = \Phi\xi \\ \Phi(z = z_0) = \phi_0 \end{cases}$$

Let $(F, B) \in \Lambda SU(2) \times \Lambda^+_{\mathbb{R}} SL(2,\mathbb{C})$ be the Iwasawa decomposition of Φ. Fix two Sym-points $\lambda_1 \neq \lambda_2 \in \mathbb{S}^1$. Define $f : \Sigma \to SU(2) \sim \mathbb{S}^3$ by the Sym-Bobenko formula

$$f = F_1\, F_2^{-1} \quad \text{with} \quad F_i(z) = F(z, \lambda_i).$$

Then

1. *If $\lambda_1 + \lambda_2 = 0$, f is a harmonic map.*
2. *If ξ_{-1} is nilpotent and does not vanish, f is a conformal immersion with constant mean curvature*
$$H = -i\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}.$$
Moreover, locally, any conformal CMC immersion into $\mathbb{S}^3$ may be obtained by this method.

Remark 2. *Many authors take ξ_{-1} upper triangular nilpotent, following [1]. This makes the formulas for the first and second fundamental forms of the immersion simpler.*

Remark 3.

In the nilpotent case, write

$$\lambda_1 = \nu e^{i\theta}, \quad \lambda_2 = \nu e^{-i\theta}$$

with $\nu \in \mathbb{S}^1$ *and* $\sin(\theta) \neq 0$. *Then*

$$H = -\cot(\theta).$$

1. *Fixing* θ *and varying* ν *produces a 1-parameter family of isometric CMC immersions with the same mean curvature. This is called the associate family.*

2. *Fixing* ν *and varying* θ *produces a 1-parameter of CMC immersions, which scaled by* $\frac{1}{\sin(\theta)}$, *are all isometric with mean curvature* $H = -\cos(\theta)$ *in the 3-sphere* $\mathbb{S}^3(\sin^2\theta)$. *This is the Lawson correspondence.*

3. *In the limit* $\theta \to 0$, *this rescaled immersion converges to an isometric* $H = -1$ *surface in* $T_e\mathbb{S}^3 = \mathfrak{su}(2) \sim \mathbb{R}^3$. *This is the again the Lawson correspondence. Explicitly,*

$$\lim_{\theta \to 0} \frac{1}{\sin\theta}(f_\theta - Id) = \lim_{\theta \to 0} \frac{1}{\sin\theta}\left(F(\nu e^{i\theta})F(\nu e^{-i\theta})^{-1} - 1\right) = 2i\nu \frac{\partial F}{\partial \lambda} F^{-1} \mid_{\lambda=\nu}$$

   which is precisely the Sym-Bobenko formula for CMC-1 surfaces in $\mathbb{R}^3$.

1.5 Gauges

A gauge G is a holomorphic map from Σ to $\Lambda^+ SL(2,\mathbb{C})$. Given a gauge G, we take $\widehat{\Phi} = \Phi G$. Then $\widehat{\Phi}$ and Φ define the same immersion. Indeed, if (F,B) is the Iwasawa decomposition of Φ and we decompose $(BG) \mid_{\lambda=0} = QR$ with $Q \in SU(2)$, $R \in \mathcal{T}$, then

$$\widehat{\Phi} = (FQ)(Q^{-1}BG)$$

and the Iwasawa decomposition of $\widehat{\Phi}$ is

$$\widehat{F} = FQ, \quad \widehat{B} = Q^{-1}BG.$$

Then $\widehat{f} = f$ in the Sym Bobenko formula, since Q is constant with respect to λ.

The gauged solution $\widehat{\Phi}$ satisfies $d\widehat{\Phi} = \widehat{\Phi}\widehat{\xi}$ with

$$\widehat{\xi} = G^{-1}\xi G + G^{-1}dG.$$

We denote $\xi \cdot G = \widehat{\xi}$ the gauged potential (the dot denotes a group action).

2 Elementary examples

2.1 CMC spheres

Take the simplest possible DPW potential:

$$\xi = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz$$

in $\mathbb{C}$ and the initial condition $\Phi(0) = \mathrm{Id}$. Then

$$\Phi = \begin{pmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{pmatrix}.$$

The Iwasawa decomposition is:

$$F = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & \lambda^{-1}z \\ -\lambda\overline{z} & 1 \end{pmatrix}$$

$$B = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & 0 \\ \lambda\overline{z} & 1+|z|^2 \end{pmatrix}$$

The Sym Bobenko formula at Sym points $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$ gives

$$f = \frac{1}{1+|z|^2} \big(1+\cos(2\theta)|z|^2,\ -\sin(2\theta)|z|^2,\ 2\sin(\theta)\mathrm{Im}(z),\ -2\sin(\theta)\mathrm{Re}(z)\big)$$

Then with $a = (\cos(\theta), -\sin(\theta), 0, 0)$, one has $\langle f, a\rangle = \cos(\theta)$, so the image is in the intersection of $\mathbb{S}^3$ with the 3-plane $\langle x, a\rangle = \cos(\theta)$: a round 2-sphere.

The potential has a pole at ∞, but the immersion of course extends smoothly at ∞. This may be seen using the following gauge:

$$G = \begin{pmatrix} z & 0 \\ -\lambda & z^{-1} \end{pmatrix}.$$

Then

$$\xi \cdot G = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2}$$

is holomorphic at ∞. We say that ∞ is an apparent singularity and G is a regularizing gauge. Apparent singularities are important for the global DPW method on compact Riemann surfaces.

2.2 CMC tori

Take

$$\xi = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} dz$$

in $\mathbb{C}$ with the initial condition $\Phi(0) = \mathrm{Id}$. Then

$$\Phi = \exp \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} z$$

$$F = \exp \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} (\lambda^{-1} z - \overline{z})$$

$$B = \exp \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \overline{z}$$

(F and B commute in this case.) The Sym Bobenko formula at $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$ gives (after a rotation of angle $\theta/2$ in the (x_1, x_2) plane)

$$f(x+iy) = (\cos(\theta/2)\cos(\widetilde{x}),\ \sin(\theta/2)\cos(\widetilde{y}),\ \sin(\theta/2)\sin(\widetilde{y}),\ -\cos(\theta/2)\sin(\widetilde{x}))$$

with $\widetilde{x} = 4\sin(\theta/2)x$ and $\widetilde{y} = 4\cos(\theta/2)y$. Up to permutation of the coordinates, the image is the torus

$$\mathbb{S}^1(\cos(\theta/2)) \times \mathbb{S}^1(\sin(\theta/2)).$$

In the minimal case $\theta = \pi/2$, this is the Clifford torus.

3 Proof: from DPW potential to CMC immersion

In this section, we prove that given a DPW potential ξ, the DPW method produces a harmonic map / conformal CMC immersion f. The loop group decomposition is only used to prove that the Maurer Cartan 1-form ω^λ has a very specific form, from which all the properties of f follow.

3.1 The Maurer Cartan 1-form ω^λ

Let F be the unitary frame produced by the DPW method. Consider the Maurer Cartan form $\omega = F^{-1}dF$. Taking the exterior derivative we obtain the Maurer Cartan equation

$$d\omega + \omega \wedge \omega = 0$$

which is the integrability condition for the differential equation $dF = F\omega$ by Frobenius theorem. Decompose ω in Fourier series for $\lambda \in \mathbb{S}^1$:

$$\omega = \sum_{k\in\mathbb{Z}} \lambda^k \omega_k.$$

Since $F \in \Lambda SU(2)$, $\omega \in \Lambda\mathfrak{s}u(2)$ so

$$\omega_{-k} = -\omega_k^*.$$

On the other hand, by differentiation of $\Phi = FB$ we obtain

$$F^{-1}dF = B\xi B^{-1} - dB\,B^{-1}.$$

Hence $\omega_k = 0$ for $k \leq -2$ and

$$\omega_{-1} = B_0\xi_{-1}B_0^{-1}dz$$

is a 1-form of type $(1,0)$. Since $\omega \in \Lambda\mathfrak{s}u(2)$ we may write

$$\omega = F^{-1}dF = (\lambda^{-1}\alpha + \beta)dz - (\lambda\alpha^* + \beta^*)d\overline{z} \tag{1}$$

with

$$\alpha = B_0\xi_{-1}B_0^{-1}, \quad \beta^* = B_{0\,\overline{z}}B_0^{-1}$$

and we see that ω is in fact defined for all $\lambda \in \mathbb{C}^*$. The Maurer Cartan equation now reads

$$-\lambda^{-1}\alpha_{\overline{z}} - \beta_{\overline{z}} - \lambda(\alpha_{\overline{z}})^* - (\beta_{\overline{z}})^* - \lambda^{-1}[\alpha,\beta^*] - \lambda[\alpha^*,\beta] - [\alpha,\alpha^*] - [\beta,\beta^*] = 0.$$

This gives the equations

$$\begin{aligned} \alpha_{\overline{z}} &= -[\alpha,\beta^*]. \\ \beta_{\overline{z}} + (\beta_{\overline{z}})^* &= -[\alpha,\alpha^*] - [\beta,\beta^*]. \end{aligned} \tag{2}$$

Remark 4. *1. The purpose of the loop group method is to produce a family of unitary 1-forms $(\omega^\lambda)_{\lambda\in\mathbb{C}^*}$ having the special form* (1) *and satisfying the Maurer Cartan equation. Once this is achieved, all the properties of f follow, with no further reference to loop groups. The loop group method comes from the world of integrable systems.*

2. A more intrinsic way to understand ω is to consider the family of flat connections $\nabla^\lambda = d + \omega^\lambda$ on the trivial bundle $\Sigma \times \mathbb{C}^2$. Indeed, if we gauge Φ by G, we have $\widehat{F} = FQ$ so $\widehat{\omega} = Q^{-1}\omega Q + Q^{-1}dQ$. This is precisely how a connection transforms under a change of gauge (meaning it is the same connection expressed in a different trivialization.) The connections ∇^λ are unitary for $\lambda \in \mathbb{S}^1$ and the Maurer Cartan equations says they are flat. The existence of such connections is equivalent to Hitchin's self-duality equations [3].

3.2 First fundamental form

For $\lambda \in \mathbb{S}^1$, we have

$$F_z = F(\overline{\lambda}\alpha + \beta), \quad F_{\overline{z}} = -F(\lambda\alpha^* + \beta^*)$$

This gives

$$f_z = (\overline{\lambda}_1 - \overline{\lambda}_2)F_1\alpha F_2^{-1}, \quad f_{\overline{z}} = -(\lambda_1 - \lambda_2)F_1\alpha^* F_2^{-1}.$$

Writing $\langle\cdot,\cdot\rangle$ for the $\mathbb{C}$-bilinear extension of the scalar product to $\mathbb{C}^4$, we have

$$\begin{aligned}\langle f_z, f_z\rangle &= \frac{1}{4}\left(\|f_x\|^2 - \|f_y\|^2 - 2i\langle f_x, f_y\rangle\right)\\ &= \det(f_z) = (\overline{\lambda}_1 - \overline{\lambda}_2)^2\det(\alpha).\end{aligned}$$

$$\begin{aligned}\langle f_z, f_{\overline{z}}\rangle &= \frac{1}{4}\left(\|f_x\|^2 + \|f_y\|^2\right)\\ &= -|\lambda_1 - \lambda_2|^2\langle F_1\alpha F_2^{-1}, F_1\alpha^* F_2^{-1}\rangle\\ &= -|\lambda_1 - \lambda_2|^2\langle \alpha, \alpha^*\rangle\\ &= \frac{1}{2}|\lambda_1 - \lambda_2|^2\mathrm{tr}(\alpha\alpha^*).\end{aligned}$$

So f is conformal $\Leftrightarrow \det(\alpha) = 0 \Leftrightarrow \det(\xi_{-1}) = 0$ and in that case, its conformal factor μ is given by

$$\mu^2 = \|f_x\|^2 = \|f_y\|^2 = |\lambda_1 - \lambda_2|^2\mathrm{tr}(\alpha\alpha^*) = 4\sin(\theta)^2\mathrm{tr}(\alpha\alpha^*). \tag{3}$$

Remark 5. *We see that if we scale by $\frac{1}{\sin\theta}$, all immersions obtained by varying θ are indeed isometric (see remark 3).*

3.3 Harmonic map

A map $f : \Sigma \to \mathbb{S}^3$ is harmonic if $\Delta f + \|df\|^2 f = 0$. We have

$$\Delta f + \|df\|^2 f = 4\left(f_{z\overline{z}} + \langle f_z, f_{\overline{z}}\rangle f\right)$$

We compute

$$\begin{aligned}f_{z\overline{z}} &= (\overline{\lambda}_1 - \overline{\lambda}_2)\left(-F_1(\lambda_1\alpha^* + \beta^*)\alpha F_2^{-1} - F_1[\alpha, \beta^*]F_2^{-1} + F_1\alpha(\lambda_2\alpha^* + \beta)F_2^{-1}\right)\\ &= (\overline{\lambda}_1 - \overline{\lambda}_2)F_1(-\lambda_1\alpha^*\alpha + \lambda_2\alpha\alpha^*)F_2^{-1}.\end{aligned} \tag{4}$$

Assume that $\lambda_1 + \lambda_2 = 0$. Then

$$f_{z\bar{z}} = -2\, F_1(\alpha\alpha^* + \alpha^*\alpha)F_2^{-1}$$

$$\langle f_z, f_{\bar{z}}\rangle f = 2\,\mathrm{tr}(\alpha\alpha^*)F_1F_2^{-1}.$$

A simple computation shows that for any $\alpha \in \mathfrak{sl}(2,\mathbb{C})$

$$\alpha\alpha^* + \alpha^*\alpha = \mathrm{tr}(\alpha\alpha^*)\mathrm{Id}$$

which gives that f is harmonic.

Remark 6. *If $\lambda_1 + \lambda_2 = 0$ and $\det(\xi_{-1}) = 0$, f is harmonic and conformal so minimal. We compute the mean curvature in the case $\lambda_1 + \lambda_2 \neq 0$ in the next section.*

3.4 Mean curvature

In this section, we assume that ξ_{-1} is nilpotent, so f is conformal. The normal vector is

$$N = \mu^{-2} f_x \times f_y = -2i\mu^{-2} f_z \times f_{\bar{z}}$$

where the cross-product is computed in the tangent space to $\mathbb{S}^3$. This gives

$$N = \frac{2i}{\mathrm{tr}(\alpha\alpha^*)} F_1(\alpha \times \alpha^*)F_2^{-1} = \frac{i}{\mathrm{tr}(\alpha\alpha^*)} F_1[\alpha, \alpha^*]F_2^{-1}. \tag{5}$$

The mean curvature is

$$\begin{aligned}
H &= \frac{1}{2\mu^2}\langle \Delta f, N\rangle \\
&= \frac{2}{\mu^2}\langle f_{z\bar{z}}, N\rangle \\
&= \frac{2i}{|\lambda_1 - \lambda_2|^2(\mathrm{tr}(\alpha\alpha^*))^2}(\overline{\lambda}_1 - \overline{\lambda}_2)\langle -\lambda_1\alpha^*\alpha + \lambda_2\alpha\alpha^*, [\alpha, \alpha^*]\rangle \quad \text{using (3),(4),(5)} \\
&= \frac{-i}{(\lambda_1 - \lambda_2)(\mathrm{tr}(\alpha\alpha^*))^2}\mathrm{tr}\left((-\lambda_1\alpha^*\alpha + \lambda_2\alpha\alpha*)(\alpha\alpha^* - \alpha^*\alpha)\right) \\
&= -i\frac{(\lambda_1 + \lambda_2)\mathrm{tr}(\alpha\alpha^*\alpha\alpha^*)}{(\lambda_1 - \lambda_2)(\mathrm{tr}(\alpha\alpha^*))^2} \quad \text{using } \alpha^2 = 0.
\end{aligned}$$

Since $\det(\alpha) = 0$, Cayley Hamilton gives $(\alpha\alpha^*)^2 = \mathrm{tr}(\alpha\alpha^*)\alpha\alpha^*$, hence

$$H = -i\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}.$$

3.5 Standard DPW case

In the case ξ_{-1} is upper triangular nilpotent

$$\xi_{-1} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} \rho & \cdot \\ 0 & \rho^{-1} \end{pmatrix}$$

where $a(z)$ is holomorphic and $\rho(z) > 0$, the previous formulas simplify to the standard formulas

$$\alpha = \begin{pmatrix} 0 & a\rho^2 \\ 0 & 0 \end{pmatrix}$$

$$\mu^2 = |\lambda_1 - \lambda_2|^2 |a|^2 \rho^4$$

$$N = F_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} F_2^{-1}.$$

4 Proof: from CMC immersion to DPW potential

In this section, we explain how to obtain a (not very explicit) DPW potential ξ from a given CMC immersion f.

4.1 The 2×2 Lax pair for surfaces in $\mathbb{S}^3$

Let $f : \Sigma \to \mathbb{S}^3$ be a conformal immersion with conformal factor $\mu = \|f_x\| = \|f_y\|$, normal N and mean curvature H. (We do not assume that H is constant yet.) We have

$$\langle f_z, f_z \rangle = 0, \quad \langle f_z, f_{\overline{z}} \rangle = \frac{\mu^2}{2}$$

$$H = \frac{1}{2\mu^2} \langle \Delta f, N \rangle = \frac{2}{\mu^2} \langle f_{z\overline{z}}, N \rangle.$$

Define

$$Q = \langle f_{zz}, N \rangle$$

so Qdz^2 is the $(2,0)$ part of the second fundamental form and is called the Hopf differential. Differentiating $\langle N, f \rangle = 0$, $\langle N, f_z \rangle = 0$, $\langle N, f_{\overline{z}} \rangle = 0$ and $\langle N, N \rangle = 1$ we obtain

$$\langle N_z, f \rangle = 0, \quad \langle N_z, f_z \rangle = -Q, \quad \langle N_z, f_{\overline{z}} \rangle = -\tfrac{\mu^2}{2} H, \quad \langle N, N \rangle = 0$$

so

$$N_z = -H f_z - \tfrac{2}{\mu^2} Q f_{\overline{z}}.$$

Differentiating $\langle f_z, f\rangle = 0$, $\langle f_z, f_z\rangle = 0$, $\langle f_z, f_{\overline{z}}\rangle = \frac{\mu^2}{2}$ and $\langle f_z, N\rangle = 0$ we obtain

$$\langle f_{zz}, f\rangle = 0, \quad \langle f_{zz}, f_z\rangle = 0, \quad \langle f_{zz}, f_{\overline{z}}\rangle = \mu\mu_z, \quad \langle f_{zz}, N\rangle = Q$$

so

$$f_{zz} = \tfrac{2\mu_z}{\mu} f_z + QN.$$

The canonical basis of $\mathbb{R}^4$ is identified with the matrices

$$e_1 = \mathrm{Id}, \quad e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Since $(f, N, \mu^{-1}f_x, \mu^{-1}f_y)$ is a positive orthonormal basis, there exists a pair $(F_1, F_2) \in SU(2) \times SU(2)$, unique up to sign, such that

$$f = F_1 e_1 F_2^{-1}, \quad N = F_1 e_2 F_2^{-1}, \quad f_x = \mu F_1 e_3 F_2^{-1}, \quad f_y = \mu F_1 e_4 F_2^{-1}.$$

Remark 7. *A choice has been on the order of the vectors of the basis. It is not essential, but by sending f_x and f_y to the off-diagonal vectors e_3 and e_4, we obtain in the end ξ_{-1} upper-triangular nilpotent: a standard DPW potential.*

Then

$$f_z = \mu F_1 e^+ F_2^{-1}, \quad f_{\overline{z}} = \mu F_1 e^- F_2^{-1}$$

with

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Consider the Maurer Cartan forms $\omega_i = F_i^{-1}dF_i$ for $i = 1, 2$ and define the matrices U_i, V_i (called a Lax pair) by

$$\omega_i = F_i^{-1}dF_i = U_i dz + V_i d\overline{z}.$$

The goal is to compute U_i, V_i in term of μ, H and Q. Since $\omega_i \in \mathfrak{su}(2)$ we have

$$V_i = -U_i^*, \quad \mathrm{tr}(U_i) = 0.$$

By differentiation

$$\begin{cases} f_z = F_1(U_1 - U_2)F_2^{-1} \\ N_z = F_1(U_1 e_2 - e_2 U_2)F_2^{-1} \\ f_{zz} = F_1(\mu_z e^+ + \mu U_1 e^+ - \mu e^+ U_2)F_2^{-1} \end{cases}$$

Comparing with the previous expressions for f_z, N_z and f_{zz}, we obtain a system of linear equations in the coefficients of U_1, U_2 which easily yields

$$U_1 = \begin{pmatrix} \frac{\mu_z}{2\mu} & \frac{\mu}{2}(1-iH) \\ \frac{-iQ}{\mu} & -\frac{\mu_z}{2\mu} \end{pmatrix}, \quad U_2 = \begin{pmatrix} \frac{\mu_z}{2\mu} & \frac{\mu}{2}(-1-iH) \\ \frac{-iQ}{\mu} & -\frac{\mu_z}{2\mu} \end{pmatrix}.$$

Hence

$$V_1 = \begin{pmatrix} -\frac{\mu_{\overline{z}}}{2\mu} & \frac{-i\overline{Q}}{\mu} \\ \frac{\mu}{2}(-1-iH) & \frac{\mu_{\overline{z}}}{2\mu} \end{pmatrix}, \quad V_2 = \begin{pmatrix} -\frac{\mu_{\overline{z}}}{2\mu} & \frac{-i\overline{Q}}{\mu} \\ \frac{\mu}{2}(1-iH) & \frac{\mu_{\overline{z}}}{2\mu} \end{pmatrix}.$$

The Maurer Cartan forms ω_i satisfy the Maurer Cartan equation:

$$d\omega_i + \omega_i \wedge \omega_i = 0 = (-U_{i\overline{z}} + V_{iz} + [U_i, V_i])dz \wedge d\overline{z}.$$

Both $i = 1$ and $i = 2$ give the same equations:

$$\begin{cases} \frac{1}{4}\Delta \log \mu + \frac{\mu^2}{4}(1+H^2) - \frac{|Q|^2}{\mu^2} = 0 \\ Q_{\overline{z}} = \frac{\mu^2}{2} H_z \end{cases} \tag{6}$$

The second one means that H is constant iff the Hopf differential is holomorphic (Hopf theorem).

4.2 Recovering the Maurer Cartan 1-form ω^λ

Assume now that H is constant. The goal is to interpolate between ω_1 and ω_2 to obtain a 1-parameter family of 1-forms $(\omega^\lambda)_{\lambda \in \mathbb{C}^*}$ of the form (1). Observe that U_1 and U_2 are almost the same, up to a ± 1 in the upper-right coefficient. Let θ such that $H = -\cot(\theta)$. Take $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$ and define for $\lambda \in \mathbb{C}^*$

$$U = \begin{pmatrix} \frac{\mu_z}{2\mu} & \frac{i\,\mu}{2\sin\theta}\lambda^{-1} \\ \frac{-i\,Q}{\mu} & -\frac{\mu_z}{2\mu} \end{pmatrix} \quad V = \begin{pmatrix} -\frac{\mu_{\overline{z}}}{2\mu} & \frac{-i\,\overline{Q}}{\mu} \\ \frac{i\,\mu}{2\sin\theta}\lambda & \frac{\mu_{\overline{z}}}{2\mu} \end{pmatrix}.$$

Then

$$U_i = U(\lambda = \lambda_i) \quad \text{and} \quad V_i = V(\lambda = \lambda_i).$$

Define

$$\omega = \omega^\lambda = U dz + V d\overline{z}.$$

Then ω is unitary for $\lambda \in \mathbb{S}^1$ and has the special form (1). The Maurer Cartan equation $d\omega + \omega \wedge \omega = 0$ gives again (6) which does not depend on λ so is satisfied for all $\lambda \in \mathbb{C}^*$ (since we know it already is at λ_1 and λ_2).

Remark 8. *At this point, we have constructed a family of flat connections $\nabla^\lambda = d + \omega^\lambda$ for $\lambda \in \mathbb{C}^*$ which are unitary for $\lambda \in \mathbb{S}^1$.*

4.3 Recovering F, Φ and finally ξ

The next step is to extend F_1, F_2 into $F \in \Lambda SU(2)$. Fix some base point $z_0 \in \Sigma$ and choose $\varphi \in \Lambda SU(2)$ such that $\varphi(\lambda = \lambda_i) = F_i(z_0)$ for $i = 1, 2$. Solve the following Cauchy problem in the simply connected domain Σ (which is possible because $d\omega + \omega \wedge \omega = 0$)

$$\begin{cases} dF = F\omega \\ F(z_0) = \varphi. \end{cases}$$

Then $F(\lambda = \lambda_i) = F_i$ for $i = 1, 2$ and since $\omega \in \Lambda\mathfrak{su}_2$, $F \in \Lambda SU(2)$.

Next we want to find $B \in \Lambda^+ SL(2, \mathbb{C})$ such that $\Phi = FB$ is holomorphic with respect to z. We have

$$\Phi_{\overline{z}} = 0 \Leftrightarrow FVB + FB_{\overline{z}} = 0 \Leftrightarrow B_{\overline{z}} = -VB.$$

Locally, one can solve the following $\overline{\partial}$ problem:

$$\begin{cases} B_{\overline{z}} = -VB \\ B(z_0) = I_2 \end{cases}$$

Indeed, since V is analytic, we can write locally $V(z) = \widetilde{V}(z, \overline{z})$ with $\widetilde{V}(z, w)$ a holomorphic function of two variables, solve the following holomorphic ODE with unknown $\widetilde{B}(z, w)$:

$$\begin{cases} \widetilde{B}_w = -\widetilde{V}\widetilde{B} \\ \widetilde{B}(z_0) = I_2 \end{cases}$$

and take $B(z) = \widetilde{B}(z, \overline{z})$. Since V is holomorphic in $\lambda \in \mathbb{C}$, $\widetilde{B}$ and B are holomorphic functions of λ, so $B \in \Lambda^+ SL(2, \mathbb{C})$. Define $\Phi = FB$ and

$$\xi = \Phi^{-1} d\Phi = B^{-1}\omega B + B^{-1} dB.$$

Then ξ is holomorphic with respect to z (since Φ is) and, given the shape of ω, is meromorphic with respect to λ with at most a simple pole at 0. Moreover, ξ_{-1} is nilpotent, so ξ is a DPW potential (note that ξ_{-1} has the standard upper triangular form.) Finally, since from the very beginning we have $f = F_1 F_2^{-1}$, the DPW method will recover the immersion f we started with.

5 Global aspects of the DPW method

5.1 The monodromy problem

Assume Σ is not simply connected. Let $\pi : \widetilde{\Sigma} \to \Sigma$ be its universal cover. We still denote ξ the lift $\pi^*\xi$ of ξ to $\widetilde{\Sigma}$. The DPW method produces a CMC immersion $f : \widetilde{\Sigma} \to \mathbb{S}^3$. On what condition does f descend to a well-defined immersion on Σ?

Let $\mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$ be the group of Deck transformations of the universal cover, namely the maps $\sigma : \widetilde{\Sigma} \to \Sigma$ such that $\pi \circ \sigma = \pi$. Let $\Phi : \widetilde{\Sigma} \to \Lambda SL(2,\mathbb{C})$ be a solution of $d\Phi = \Phi\xi$. Let $\sigma \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. Since $\sigma^*\xi = \xi$,

$$\Phi \circ \sigma = \mathcal{M}(\Phi, \sigma)\Phi$$

for some matrix $\mathcal{M}(\Phi, \sigma) \in \Lambda SL(2,\mathbb{C})$ independent of z, called the monodromy of Φ.

Proposition 1. *Assume that for all $\sigma \in$ Deck$(\widetilde{\Sigma}/\Sigma)$*

$$\begin{cases} \mathcal{M}(\Phi, \sigma) \in \Lambda SU(2) \\ \mathcal{M}(\Phi, \sigma) \mid_{\lambda=\lambda_1} = \mathcal{M}(\Phi, \sigma) \mid_{\lambda=\lambda_2} = \pm \mathrm{Id}. \end{cases} . \tag{7}$$

Then the immersion produced by the DPW method is well-defined on Σ.

Proof: the first condition tells us that

$$F \circ \sigma = \mathcal{M}(\Phi, \sigma)F, \quad B \circ \sigma = B$$

by uniqueness of the Iwasawa decomposition. The second condition ensures that $f \circ \sigma = f$, so f descends to a well-defined immersion on Σ. $\square$

Remark 9. (7) *is called the monodromy problem. It is the DPW analogue of the period problem for constructing minimal surfaces in $\mathbb{R}^3$ via Weierstrass representation. Note that the monodromy problem is a sufficient condition but is not necessary: for the torus example (section 2.2), the monodromy problem is not solved but the immersion is well-defined in the quotient of $\mathbb{R}^2$ by a suitable lattice.*

5.2 Energy / area

In this section, we prove that on a compact Riemann surface, the energy of the harmonic map f produced by the DPW method can be computed as certain residues. In the minimal case, this formula computes the area, and in the CMC case, the Willmore energy.

In the harmonic case ($\lambda_1 + \lambda_2 = 0$), the energy of f is

$$\begin{aligned}
\mathcal{E}(f) &= \tfrac{1}{2}\iint_\Sigma |df|^2 dx \wedge dy \\
&= i\iint_\Sigma \langle f_z, f_{\bar{z}}\rangle dz \wedge d\bar{z} \\
&= 2i\iint_\Sigma \operatorname{tr}(\alpha\alpha^*) dz \wedge d\bar{z}.
\end{aligned}$$

In the conformal case (ξ_{-1} nilpotent), the Willmore energy of f is

$$\begin{aligned}
\mathcal{W}(f) &= \iint_\Sigma (1 + H^2)\mu^2 dx \wedge dy \\
&= \iint_\Sigma \left(1 + \frac{\cos^2\theta}{\sin^2\theta}\right) 4\sin^2\theta \operatorname{tr}(\alpha\alpha^*) \frac{dz \wedge d\bar{z}}{-2i} \\
&= 2i\iint_\Sigma \operatorname{tr}(\alpha\alpha^*) dz \wedge d\bar{z}
\end{aligned}$$

Of course, in the minimal case, both formulas give the area. We will denote the energy/Willmore energy/area by $\mathcal{E}(f)$ in all cases.

Theorem 3. *[2] Let Σ be a compact Riemann surface and ξ a DPW potential with n apparent sinularities at $p_1, \cdots, p_n$ and assume that the solution Φ has unitary monodromy. Then*

$$\mathcal{E}(f) = 4\pi \sum_{j=1}^{n} \operatorname{Res}_{p_j} \operatorname{tr}\left(\xi_{-1} G_1^j (G_0^j)^{-1}\right).$$

where $G^1, \cdots, G^n$ are regularizing local gauges at $p_1, \cdots, p_n$ (and as usual, an index k denotes the coefficient of λ^k).

Remark 10. *As a consequence, on a compact Riemann surface, apparent singularities are necessary if we are to solve the monodromy problem.*

Example: in the case of a round sphere with an apparent singularity at ∞, we obtain with the gauge given in section 2.1

$$\operatorname{tr}(\eta^{-1} G_1 G_0) = \operatorname{tr}\begin{pmatrix} 0 & dz \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = \frac{-dz}{z}$$

which gives $\mathcal{W}(f) = 4\pi$.

The theorem follows from the following two lemmas.

Lemma 1. *Let $\Omega \subset \Sigma$ be a smooth domain where ξ is holomorphic. Then*

$$\mathcal{E}(f(\Omega)) = -2i \int_{\partial\Omega} \mathrm{tr}(\xi_{-1} B_0^{-1} B_1).$$

Proof: first of all, if Φ has unitary monodromy, its positive part B is well-defined on Σ. We have from (1)

$$\alpha dz = B_0 \xi_{-1} B_0^{-1}, \quad \alpha^* d\overline{z} = (\overline{\partial} B\, B^{-1})_1 = \overline{\partial} B_1\, B_0^{-1} - \overline{\partial} B_0\, B_0^{-1}\, B_1\, B_0^{-1}$$

$$\mathrm{tr}(\alpha\alpha^*) dz \wedge d\overline{z} = \mathrm{tr}\left(\cancel{B_0}\, \xi_{-1}\, B_0^{-1} \wedge \overline{\partial} B_1\, \cancel{B_0^{-1}}\right) - \mathrm{tr}\left(\cancel{B_0}\, \xi_{-1}\, B_0^{-1} \wedge \overline{\partial} B_0\, B_0^{-1}\, B_1\, \cancel{B_0^{-1}}\right)$$

$$\begin{aligned} d\,\mathrm{tr}(\xi_{-1} B_0^{-1} B_1) &= \mathrm{tr}(-\xi_{-1} \wedge \overline{\partial}(B_0^{-1} B_1) \\ &= \mathrm{tr}\left(\xi_{-1} \wedge B_0^{-1}\, \overline{\partial} B_0^{-1}\, B_0^{-1}\, B_1\right) - \mathrm{tr}\left(\xi_{-1} \wedge B_0^{-1}\, \overline{\partial} B_1\right) \end{aligned}$$

Hence

$$\mathrm{tr}(\alpha\alpha^*) dz \wedge d\overline{z} = -d\,\mathrm{tr}(\xi_{-1} B_0^{-1} B_1).$$

The lemma follows from Stokes theorem. □

Lemma 2. *Let p be an apparent singularity and G a gauge such that $\xi \cdot G$ extends holomorphically to the disk $D(p,r)$. Then*

$$\lim_{r \to 0} \int_{\partial D(p,r)} \mathrm{tr}(\xi_{-1} B_0^{-1} B_1) = -2\pi i\, \mathrm{Res}_p \mathrm{tr}(\xi_{-1} G_1 G_0^{-1}).$$

Proof: with the notations of section 1.5

$$\xi_{-1} = G_0 \widehat{\xi}_{-1} G_0^{-1}$$

$$\widehat{F} = FQ, \quad \widehat{B} = Q^{-1} B G$$

$$B_0 = Q \widehat{B}_0 G_0^{-1}$$

$$B_1 = Q \widehat{B}_1 G_0^{-1} - Q \widehat{B}_0 G_0^{-1} G_1 G_0^{-1}$$

$$\mathrm{tr}(\xi_{-1} B_0^{-1} B_1) = \mathrm{tr}\left(G_0 \widehat{\xi}_{-1} \widehat{B}_0^{-1} (\widehat{B}_1 G_0^{-1} - \widehat{B}_0 G_0^{-1} G_1 G_0^{-1}\right)$$

$$\int_{\partial D(p,r)} \mathrm{tr}(\xi_{-1} B_0^{-1} B_1) = \int_{\partial D(p,r)} \mathrm{tr}(\widehat{\xi}_{-1} \widehat{B}_0^{-1} \widehat{B}_1) - \int_{\partial D(p,r)} \mathrm{tr}(\widehat{\xi}_{-1} G_0^{-1} G_1).$$

The first integral goes to 0 as $r \to 0$ because the integrant is smooth in the the disk $D(p,r)$. The second one evaluates as a residue because the integrant is meromorphic. □

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