Lawson's Bipolar Minimal Surfaces in the 5-Sphere

(joint work with E. Mäder-Baumdicker)

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Outline

Motivation

Closed Minimal Surfaces in Spheres

Unlike $\mathbb{R}^n$ as an ambient manifold, the n-sphere $\mathbb{S}^n$, $n \geq 3$, allows for *closed* minimal surfaces.

The latter, in particular their *topological classification*, turned out to be highly relevant

- in the geometric optimization of Laplacian eigenvalues on closed surfaces,
- for the Willmore problem.

The Willmore Functional

What is the *best* shape an immersed closed surface $f\colon \Sigma \to \mathbb{R}^n$ $(n \geq 3)$ with fixed topology can take in the ambient space?

A natural quantity to employ for this issue is the Willmore functional

$$W[f] := \int_{\Sigma} |H|^2 \, \mathrm{d}\mu$$

where H is the mean curvature vector of f and $\mathrm{d}\mu$ its induced area measure.

The Willmore Problem

... comprises the search for minimizers of the Willmore energy restricting to surfaces $f : \Sigma \to \mathbb{R}^n$ with *fixed topology*. Only few is known on explicit minimizers:

- Round spheres are the only closed surfaces with $W[f] = 4\pi$. Every other closed immersed surface in $\mathbb{R}^n$, $n \geq 3$, satisfies $W[f] > 4\pi$.
- $W[f] \geq 2\pi^2$ for orientable surfaces of genus $g \geq 1$ in $\mathbb{R}^3$, where equality holds for stereographic projections of the Clifford torus in $\mathbb{S}^3$.
- $W[f] \geq 6\pi$ for $f : \mathbb{R}P^2 \to \mathbb{R}^n$, $n \geq 3$, where equality holds for stereographic projections of the Veronese surface in $\mathbb{S}^4$.

The Willmore Problem

For higher genera and higher codimensions, the Willmore problem is open.

- For orientable genus $g \geq 2$, there exist smooth minimizers with energy $< 8\pi$.
- Surfaces with self-intersections satisfy $W[f] \geq 8\pi$ and hence can be ruled out as potential minimizers.

And there are the **conjectures**:

- ... Lawson surfaces $\xi_{g,1} \subseteq \mathbb{S}^3$ are the minimizers in $\mathbb{R}^3$ for orientable genus $g \geq 2$.
- ... bipolar Lawson surface $\widetilde{\tau}_{3,1} \subseteq \mathbb{S}^4$ are the minimizers among Klein bottles in $\mathbb{R}^n$, $n \geq 4$.

Minimal Surfaces in Spheres provide Willmore Surfaces.

Proposition (Weiner 1978, Thomsen 1923)

Let $\Phi : \mathbb{R}^n \to \mathbb{S}^n \setminus \{-e_{n+1}\}$ be the inverse stereographic projection. Then, for any closed immersed surface $f : \Sigma \to \mathbb{R}^n$ we have

$$W[f] = \int_\Sigma \left(|H_{\Phi \circ f}|^2 + 1\right) \mathrm{d}\mu_{\Phi \circ f}.$$

In particular, closed minimal surfaces in $\mathbb{S}^n$ stereographically project onto Willmore surfaces in $\mathbb{R}^n$ and their area is the Willmore energy of the stereographic projection.

Bipolar Surfaces in the 5-Sphere

Proposition (H. B. Lawson, 1970)

Let $\psi\colon \Sigma \to \mathbb{S}^3 \subseteq \mathbb{R}^4$ *be an oriented minimal surface with Gauss map* $\nu\colon \Sigma \to \mathbb{S}^3 \subseteq \mathbb{R}^4$.

$$\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5 \subseteq \mathbb{R}^6, \quad \widetilde{\psi} := \psi \wedge \nu$$

is a minimal surface, the so-called bipolar surface of ψ. *For the induced metrics, we have*

$$g_{\widetilde{\psi}} = (2 - K) g_\psi \,,$$

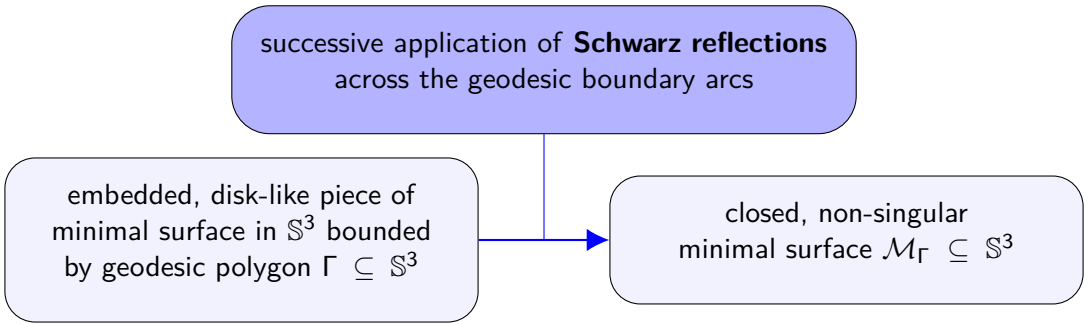
where K *is the Gaussian curvature of* (Σ, g_ψ).

In particular, if Σ is *closed*, then $\boxed{\operatorname{area}\left(\widetilde{\psi}\right) = 2\operatorname{area}(\psi) - 2\pi\chi(\Sigma)\,.}$

Lawson's Construction of Minimal Surfaces in the 3-Sphere

Construction Procedure for Minimal Surfaces in the 3-Sphere

Under certain conditions, Lawson's construction is of the following principle:

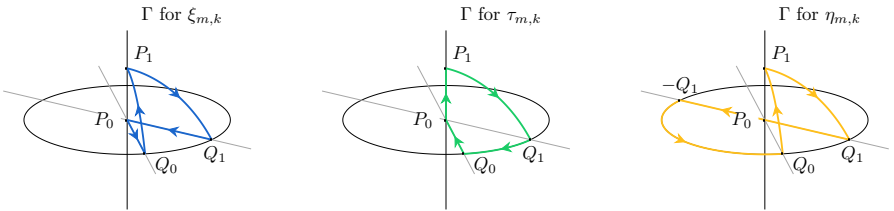


Lawson's Families of Closed Minimal Surfaces in the 3-Sphere

Explicit choices for Γ led to three infinite families

$$(\xi_{m,k})_{m,k\in\mathbb{Z}_{\geq 1}},\ (\tau_{m,k})_{m,k\in\mathbb{Z}_{\geq 1}} \text{ and } (\eta_{m,k})_{m,k\in\mathbb{Z}_{\geq 1}}$$

of highly symmetric, closed minimal surfaces in $\mathbb{S}^3$.



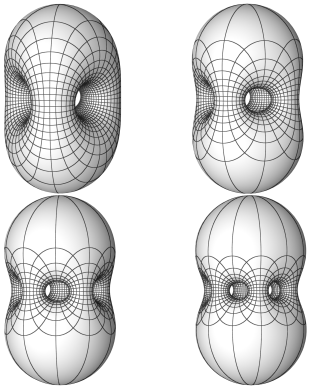
Lawson's Families of Closed Minimal Surfaces in the 3-Sphere

Concerning their topology and embeddedness the three families *differ significantly*:

- The $\xi_{m,k}$ provide embedded examples of every orientable genus ≥ 1.
- The $\tau_{m,k}$ are a family of immersed tori and Klein bottles.
- The $\eta_{m,k}$ provide immersed examples of every non-orientable genus ≥ 2.

The *Clifford torus* $\frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1 \subseteq \mathbb{S}^3$ is realized by $\xi_{1,1}$ as well as $\tau_{1,1}$.

Lawson's Families of Closed Minimal Surfaces in the 3-Sphere



The surfaces $\xi_{g,1}$, $g = 2, 3, 4, 5$, stereographically projected to $\mathbb{R}^3$

Reference: https://page.math.tu-berlin.de/∼schmitt/gallery/resources/090002.png, 11/17/2022

Lawson's Families of Closed Minimal Surfaces in the 5-Sphere

The concept of the bipolar surface in particular yields the three families

$$\left(\widetilde{\xi}_{m,k}\right)_{m,k\in\mathbb{Z}_{\geq 1}},\ \left(\widetilde{\tau}_{m,k}\right)_{m,k\in\mathbb{Z}_{\geq 1}} \text{ and } \left(\widetilde{\eta}_{m,k}\right)_{m,k\in\mathbb{Z}_{\geq 1}}$$

of closed minimal surfaces in $\mathbb{S}^5$.

What is their topology?

Analysis of Lawson's Bipolar Surfaces in the 5-Sphere

Motivation: The Bipolar τ-family

The Topology of the Surfaces $\widetilde{\tau}_{m,k}$

Theorem (H. Lapointe, 2008)

Let $m,\ k \in \mathbb{Z}_{\geq 1}$ and $\gcd(m,k) = 1$.

- *If $mk \equiv 0 \bmod 2$, then $\widetilde{\tau}_{m,k}$ is a torus.* *(Recall that $\tau_{m,k}$ is a Klein bottle.)*
- *If $mk \equiv 1 \bmod 4$, then $\widetilde{\tau}_{m,k}$ is a torus.* *(Recall that $\tau_{m,k}$ is a torus.)*
- *If $mk \equiv 3 \bmod 4$, then $\widetilde{\tau}_{m,k}$ is a Klein bottle.* *(Recall that $\tau_{m,k}$ is a torus.)*

In general, the topology of the bipolar surface can differ and its actual area can be *smaller* as the value obtained from the area formula.

Key Question

How can we determine the *topology* and the *area* of the bipolar surfaces without an explicit parametrization?

Reformulation of Lawson's Construction Procedure in $\mathbb{S}^3$

Setup of Lawson's Construction Procedure

Let

- Δ be the closed unit disk,
- $\Gamma \subseteq \mathbb{S}^3$ a geodesic polygon satisfying the Lawson's conditions,
- $f : \Delta \to \mathbb{S}^3$ an immersion of the unique embedded, minimal disk with boundary Γ,
- $G \subset \mathrm{SO}(4)$ be the group generated by the reflections across the boundary arcs of Γ,
- $G^p \subseteq G$ be the subgroup leaving $f(p)$ invariant.

We can assume that the identity is the *only* element in G leaving Γ invariant.

Immersion into $\mathbb{S}^3$ on a Smallest Possible Domain S

Since by Lawson's construction

$$\mathcal{M}_\Gamma = \bigcup_{g \in G} (g \circ f)(\Delta)\,,$$

we define on $G \times \Delta$

$$(g,p) \sim (h,q) \;:\Leftrightarrow\; (g \circ f)(p) = (h \circ f)(q) \text{ and } g^{-1} \cdot h \in G^p,$$

and obtain the manifold

$$S := (G \times \Delta)/\sim,$$

such that

$$\psi : S \to \mathbb{S}^3,\; \psi\Big([(g,p)]\Big) := (g \circ f)(p)$$

immerses the closed minimal surface $\mathcal{M}_\Gamma \subseteq \mathbb{S}^3$ *on a smallest possible domain*.

The Topology of the domain S

Proposition (Lawson, 1970)

We have

$$\chi(S) = |G| \cdot \left(1 - \sum_{i=1}^{N} \frac{k_i - 1}{2k_i}\right),$$

where $\frac{\pi}{k_1}., ..., \frac{\pi}{k_N}$, $k_i \in \mathbb{Z}_{\geq 2}$, *denote the interior angles of* Γ.

Proposition (Mäder-Baumdicker, R., 2022)

S is non-orientable if and only if the identity $e_G \in G$ *can be written as a product of an odd number of the generators* $r_{\gamma_1}, ..., r_{\gamma_N}$.

Definition of a Gauss Map

If S is orientable, we can define the parity $\sigma(g)$ of $g \in G$ by

$$\sigma(g) := \begin{cases} 0 \\ 1 \end{cases} \text{ if } g \text{ is given by products of } \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \text{ numbers of the generators } r_{\gamma_1}, ..., r_{\gamma_N}.$$

Lemma (Mäder-Baumdicker, R., 2022)

Let $n : \Delta \to \mathbb{S}^3$ be a Gauss map of f.

$$\nu : S \to \mathbb{S}^3, \ \nu([(g, p)]) := (-1)^{\sigma(g)} \cdot (g \circ n)(p)$$

is a Gauss map of $\psi : S \to \mathbb{S}^3$.

Analysis of the Bipolar Immersions

Definition of the Bipolar Immersion

If S is orientable, the bipolar surface $\widetilde{\mathcal{M}}_\Gamma \subseteq \mathbb{S}^5$ of $\mathcal{M}_\Gamma \subseteq \mathbb{S}^3$ is immersed by

$$\widetilde{\psi} := \psi \wedge \nu \colon S \to \mathbb{S}^5,$$

$$\widetilde{\psi}([(g,p)]) := (-1)^{\sigma(g)} \cdot (g \circ f)(p) \wedge (g \circ n)(p)\,,$$

possibly *not yet* on a smallest possible domain.

If S is non-orientable, we define $\widetilde{\psi} : \overline{S} \to \mathbb{S}^5$ on the orientable double cover $\overline{S}$ of S, which can be constructed similar to S.

Cover due to $-\mathbb{1}_{\mathbb{R}^4} \in G$

Theorem (Mäder-Baumdicker, R., 2022)

(i) *Suppose that S is orientable and $-\mathbb{1}_4 \in G$ with $\sigma(-\mathbb{1}_4) = 0$.*

Then, the action

$$\langle -\mathbb{1}_4 \rangle \times S \to S, \ (h, [(g, p)]) \mapsto [(hg, p)]$$

*leaves $\widetilde{\psi} : S \to \mathbb{S}^5$ invariant and induces a **covering map of degree 2** on S such that the quotient $S/\langle -\mathbb{1}_4 \rangle$ is **orientable**. In that case, we have*

$$\operatorname{area}\left(\widetilde{\mathcal{M}}_\Gamma\right) \leq \operatorname{area}(\mathcal{M}_\Gamma) - \pi\chi(S)\,.$$

Cover due to $-\mathbb{1}_{\mathbb{R}^4} \in G$

Theorem (Mäder-Baumdicker, R., 2022)

(ii) *Suppose that S is non-orientable and $-\mathbb{1}_4 \in G$.*

Then, the action of $\langle -\mathbb{1}_4 \rangle$ leaves $\widetilde{\psi} : \overline{S} \to \mathbb{S}^5$ invariant and induces a ***covering map of degree 2*** *on $\overline{S}$ such that the quotient $\overline{S}/\langle -\mathbb{1}_4 \rangle$ is* ***orientable****. In that case, we have*

$$\mathrm{area}\left(\widetilde{\mathcal{M}}_\Gamma\right) \leq 2\mathrm{area}(\mathcal{M}_\Gamma) - 2\pi\chi(S)\,.$$

Application to the Bipolar ξ- and η-Families

In the previous setup, we consider the bipolar surfaces $\widetilde{\xi}_{m,k} \subseteq \mathbb{S}^5$ and $\widetilde{\eta}_{m,k} \subseteq \mathbb{S}^5$ for $m, k \in \mathbb{Z}_{\geq 1}$ with $m > 1$ or $k > 1$ (note that $\xi_{1,1} = \tau_{1,1}$ and $\eta_{1,1} = \tau_{2,1}$).

For the corresponding surfaces in $\mathbb{S}^3$ we have

- $\xi_{m,k} \subseteq \mathbb{S}^3$ has *orientable* genus mk.
- If k is odd, $\eta_{m,k} \subseteq \mathbb{S}^3$ has *non-orientable* genus $mk + 1$.
- If k is even, $\eta_{m,k} \subseteq \mathbb{S}^3$ has *orientable* genus mk.

Characterization of the Bipolar ξ-Family

Theorem (Mäder-Baumdicker, R., 2022)

For $m,\ k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$, the bipolar surface $\tilde{\xi}_{m-1,k-1}$ is orientable, and

(i) *if both m and k are even, we have*

$$\chi\Big(\tilde{\xi}_{m-1,k-1}\Big) = 1 - (m-1)(k-1)\,;$$

$$2\pi \max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\xi}_{m-1,k-1}\Big) < 2\pi\big(mk + k - m\big)\,;$$

(ii) *if m or k is odd, we have*

$$\chi\Big(\tilde{\xi}_{m-1,k-1}\Big) = 2\big(1 - (m-1)(k-1)\big)\,;$$

$$4\pi \max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\xi}_{m-1,k-1}\Big) < 4\pi\big(mk + k - m\big)\,.$$

Characterization of the Bipolar η-Family

Theorem (Mäder-Baumdicker, R., 2022)

For $m,\ k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$, the bipolar surface $\tilde{\eta}_{m-1,k-1}$ is orientable, and

(i) *if both m and k are even, we have*

$$\chi\left(\tilde{\eta}_{m-1,k-1}\right) = 1 - (m-1)(k-1)\,;$$

$$2\pi \max\{m,k\} \leq \operatorname{area}\left(\widetilde{\eta}_{m-1,k-1}\right) < 2\pi(3mk - 3k - m)\,;$$

(ii) *if m or k is odd, we have*

$$\chi\left(\tilde{\eta}_{m-1,k-1}\right) = 2(1 - (m-1)(k-1))\,;$$

$$4\pi \max\{m,k\} \leq \operatorname{area}\left(\widetilde{\eta}_{m-1,k-1}\right) < 4\pi(3mk - 3k - m)\,.$$

Non-Embeddedness of the Bipolar ξ- and η-Families

As we detected *transversally intersecting tangent planes* in each case, we find:

Corollary (Mäder-Baumdicker, R., 2022)

For $m,\ k \in \mathbb{Z}_{\geq 1}$ with $m > 1$ or $k > 1$, the surfaces $\widetilde{\xi}_{m,k} \subseteq \mathbb{S}^5$ and $\widetilde{\eta}_{m,k} \subseteq \mathbb{S}^5$ are not embedded.

Conclusion and Open Questions

Conclusion and Open Questions

The bipolar $\widetilde{\xi}$- and $\widetilde{\eta}$-families in $\mathbb{S}^5$ do *not* yield comparison surfaces for the Willmore problem with Willmore energy below 8π.

However, the Klein bottle $\widetilde{\tau}_{3,1}$ motivates a further investigation of bipolar surfaces. Open questions are for example:

- Do other examples of closed minimal surfaces in $\mathbb{S}^3$ lead to useful bipolar surfaces?
- In general, which properties of closed minimal surfaces in $\mathbb{S}^3$ lead to embedded (non-orientable) bipolar surfaces?

Thanks for your attention.

References

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