$\mathcal{H}$-workshop 2023

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Bernstein type results for Elliptic Weingarten surfaces

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Weingarten Surfaces

S is a Weingarten surface (**W-surface**) if there is a relation between its curvatures:

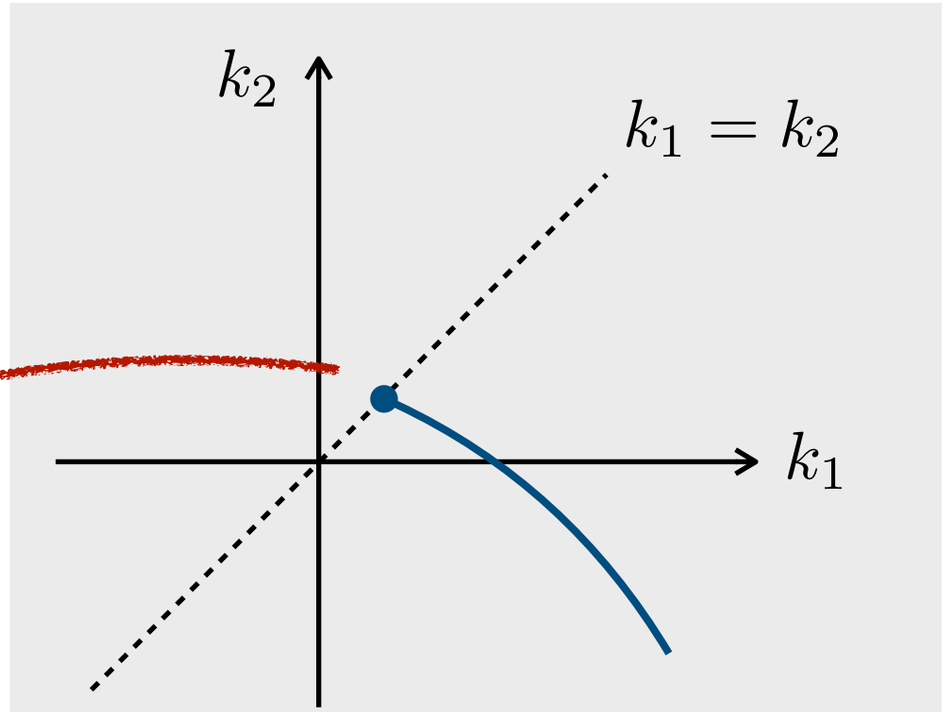
$$W(k_1, k_2) = 0 \quad \text{with} \quad W \in \mathcal{C}^1(\mathbb{R}^2) \quad \text{symmetric} \qquad (\text{or} \quad F(H, K) = 0\ )$$

and it is an Elliptic Weingarten (**EW**) surface if the underlying PDE is *elliptic*:

$$W_{k_1} W_{k_2} > 0 \quad \text{on } W^{-1}(0)$$

or, equivalently

$$k_2 = f(k_1) \text{ with } f' < 0$$



$k_1 \geq k_2$ principal curvatures

$\exists_1 \alpha$ such that $f(\alpha) = \alpha$.

- $\alpha = 0 \rightsquigarrow$ of minimal type (includes planes)
- $\alpha \neq 0 \rightsquigarrow$ of non-minimal type (includes spheres)

Weingarten Surfaces

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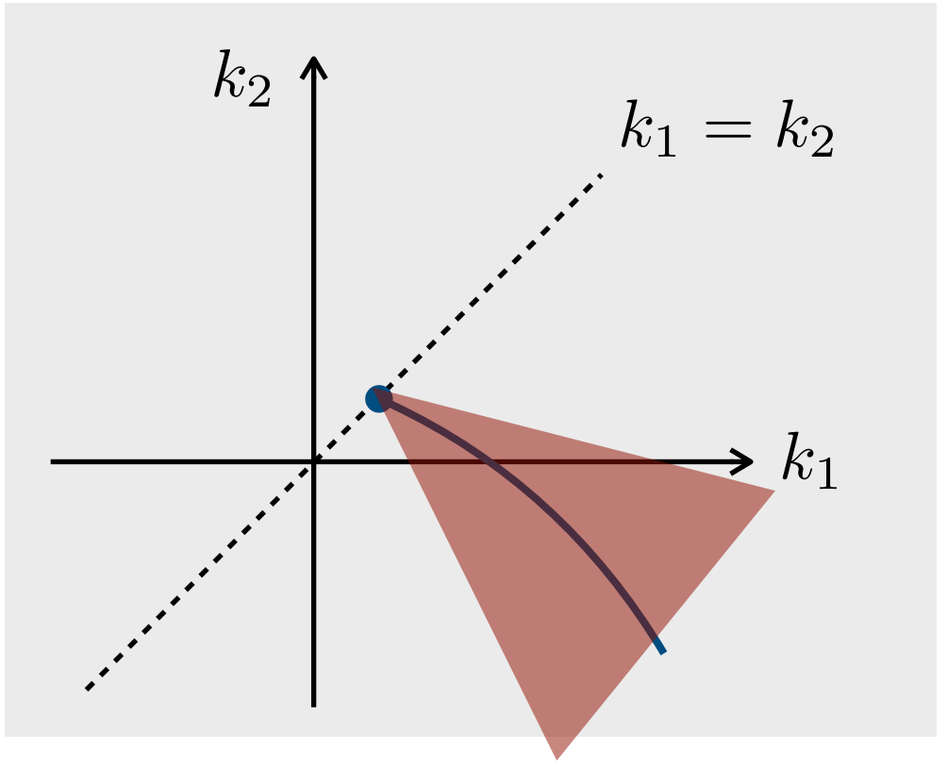
Uniformly Elliptic Weingarten:

$$\Lambda_1 < f' < \Lambda_2 < 0$$



$k_1 \geq k_2$ principal curvatures

Some Particular Theories

EW = Elliptic Weingarten:

$k_2 = f(k_1)$ with $f' < 0$

- **Constant Mean Curvature (CMC)** and **minimal** surfaces

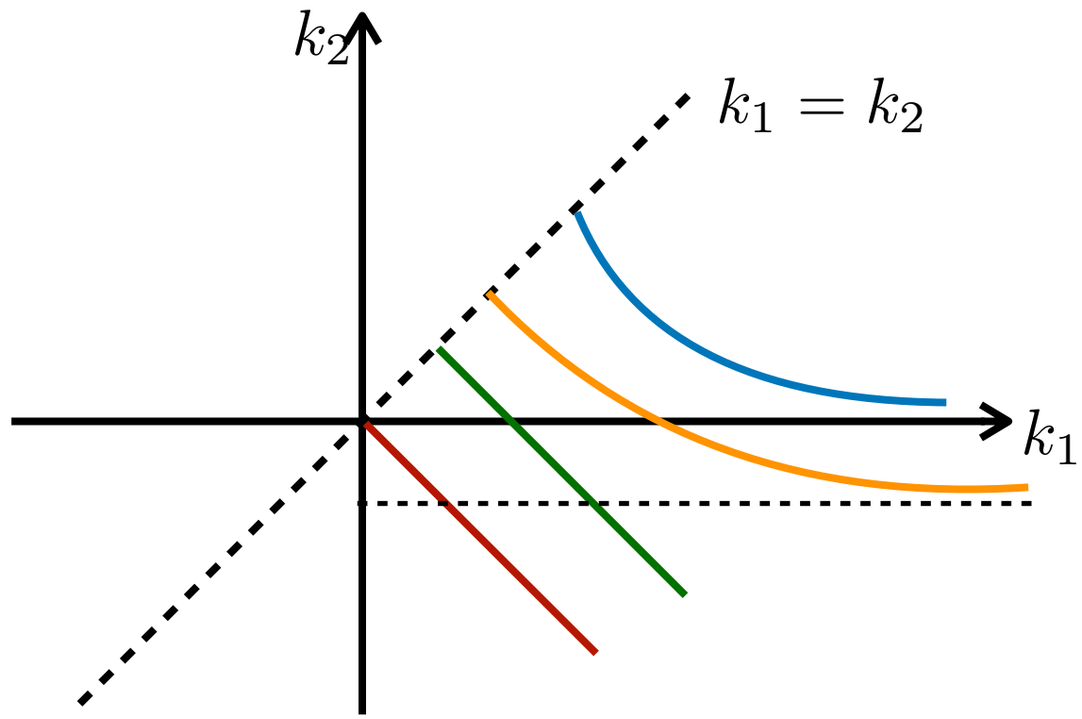
EW surfaces **extend** the theory of
minimal and CMC surfaces
(from quasilineal to fully nonlinear elliptic PDE)

- **Positive Constant Gaussian** Curvature surfaces

- **Linear elliptic Weingarten surfaces**:

$$2\alpha H + \beta K = \delta \quad \text{or} \quad k_2 = \frac{\delta - \alpha k_1}{\alpha + \beta k_1}$$

with $\alpha^2 + \beta\delta > 0$ (ellipt. condit.)



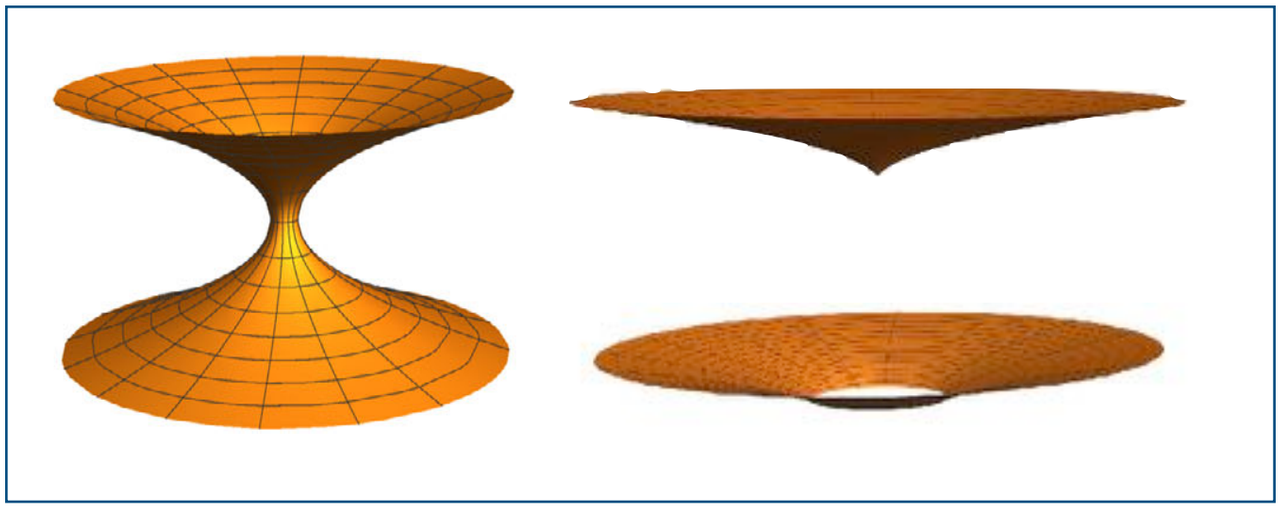
Elliptic Weingarten Surfaces - Examples

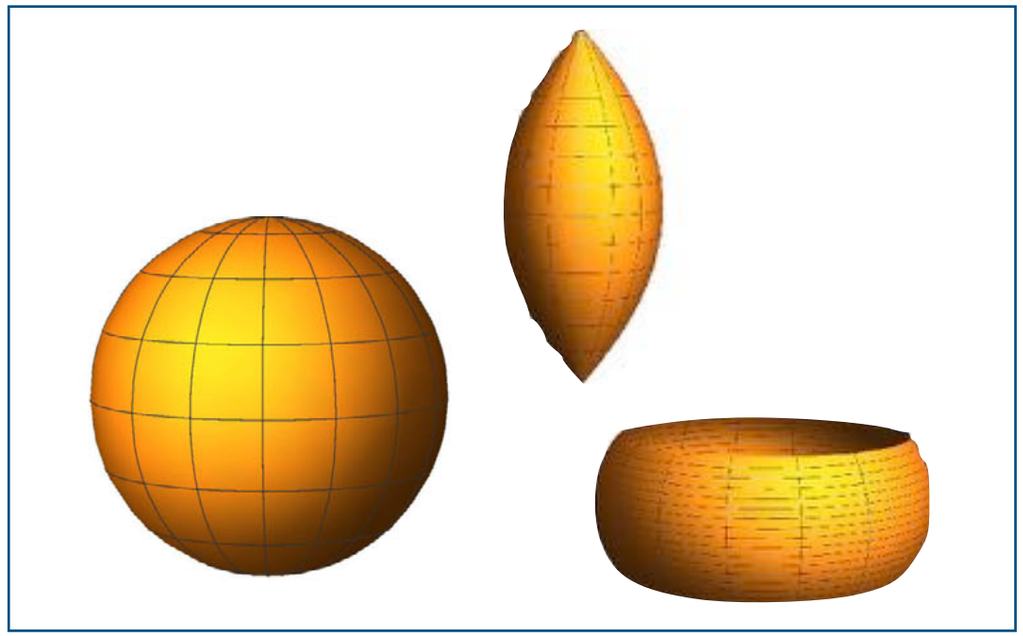
EW = Elliptic Weingarten:

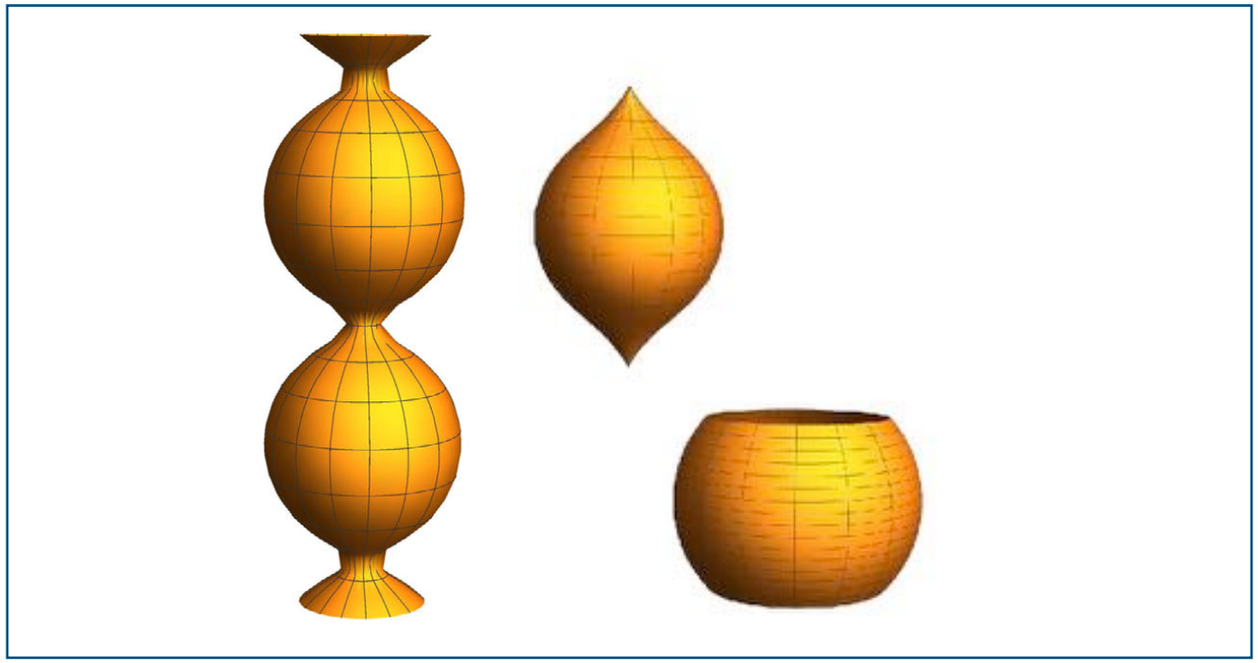
$$k_2 = f(k_1) \text{ with } f' < 0$$

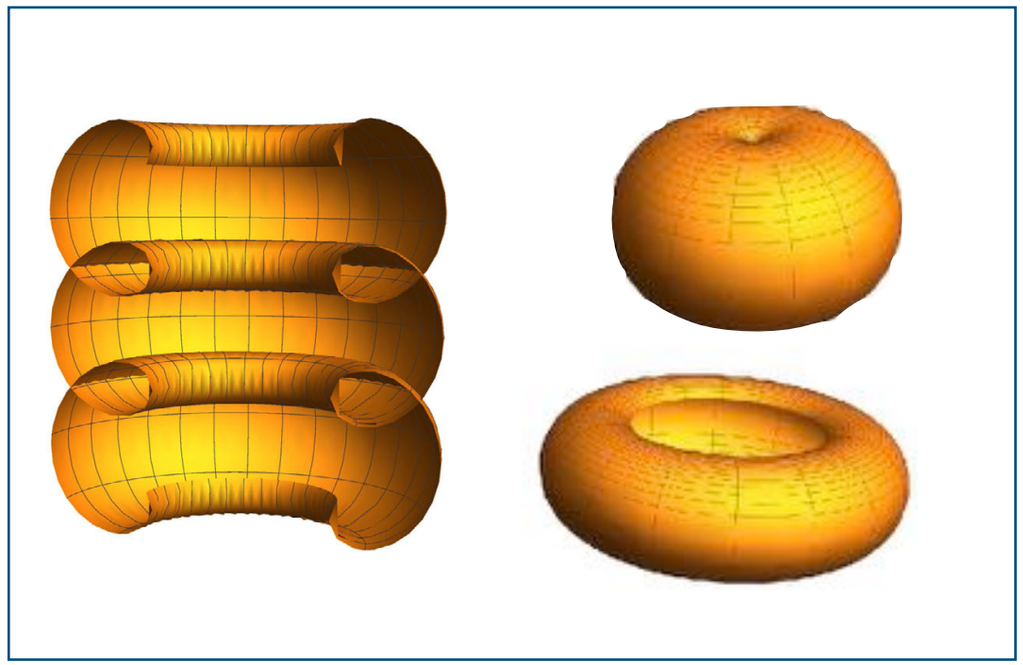
Complete rotational EW surfaces were classified by SaEarp&Toubiana (1999):

Non complete rotational EW surfaces were classified by —&Mira (2022):







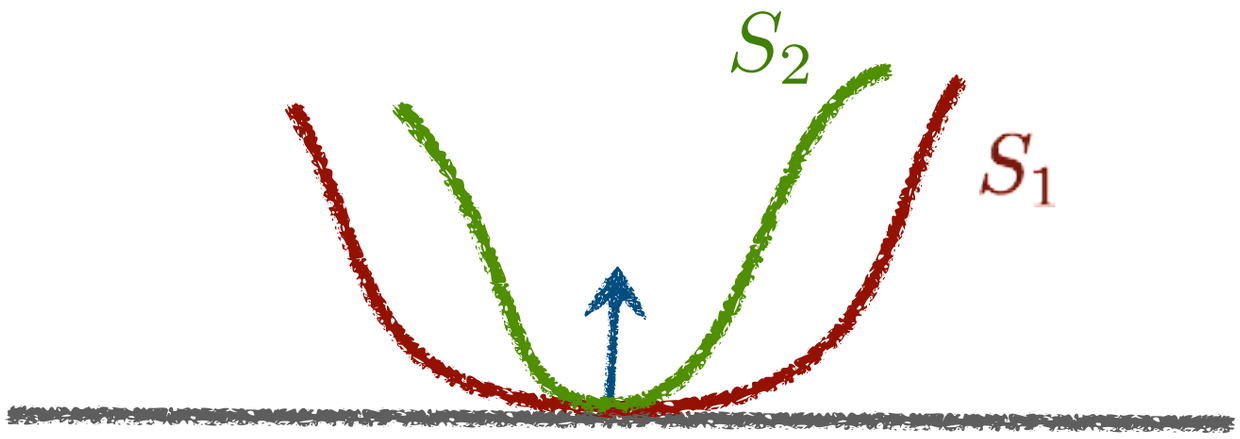


GLOBAL GEOMETRY OF EW-SURFACES

EW = Elliptic Weingarten:

$k_2 = f(k_1)$ with $f' < 0$

- EW-surfaces satisfy an elliptic fully nonlinear PDE and, as a consequence, a MAXIMUM PRINCIPLE.



Two EW-surfaces that are tangent at one point cannot lie one *above* the other (unless they are the same surface!)

GLOBAL GEOMETRY OF EW-SURFACES

EW = Elliptic Weingarten:

$$k_2 = f(k_1) \text{ with } f' < 0$$

- EW-surfaces satisfy an elliptic fully nonlinear PDE and, as a consequence, a MAXIMUM PRINCIPLE.

- Alexandrov type theorem:
  Compact embedded EW-surfaces are *round* spheres.

- Hopf type theorem (Hopf, Chern, Hartman & Wintner, Galvez & Mira):
  Genus zero compact EW-surfaces are *round* spheres.

- Study of properly embedded ends of EW-surfaces of non-minimal type (Rosenberg & SaEarp, 1994)

- Half-space theorem for EW-surfaces of minimal type (SaEarp & Toubiana, —&Mira)

- Finite total curvature theory for EW-surfaces of minimal type (Espinar & Mesa, 2019)

TWO OPEN PROBLEMS FOR EW-SURFACES

EW = Elliptic Weingarten:

$$k_2 = f(k_1) \text{ with } f' < 0$$

- Bernstein (1909): CMC (including minimal) entire graphs are planes.

BERNSTEIN PROBLEM: **(PROPOSED BY ROSENBERG & SAEARP IN 1994)**

Are planes the only EW entire graphs?

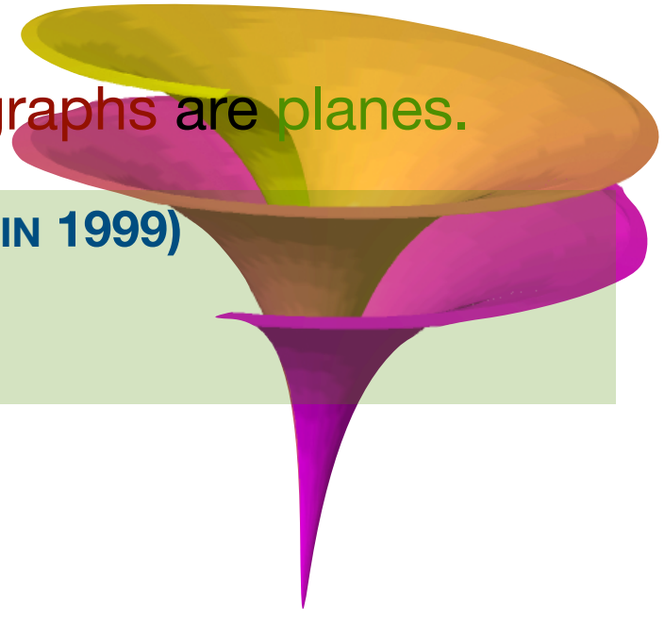
- Bernstein problem is related with the size of the Gauss map (Gauss map of graphs take values on an open hemisphere).

- MULTIGRAPH (local graph) = the Gauss map takes values on an open hemisphere.

- Hoffman & Osserman & Schoen (1982): CMC complete multigraphs are planes.

GENERALIZED BERNSTEIN PROBLEM: **(PROPOSED BY SAEARP & TOUBIANA IN 1999)**

Are planes the only EW complete multigraphs?



Aim of the Talk (and some Remarks)

EW = Elliptic Weingarten:

$k_2 = f(k_1)$ with $f' < 0$

BERNSTEIN PROBLEM: (proposed by Rosenberg & SaEarp in 1994)

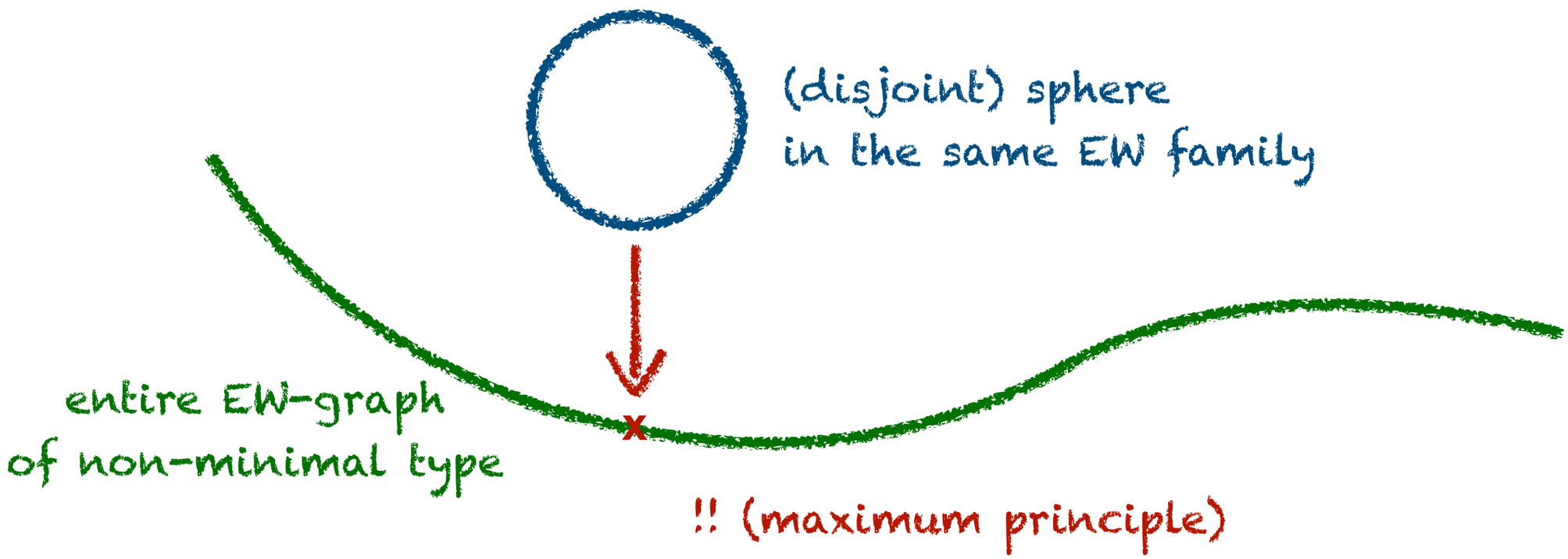
Are planes the only EW entire graphs?

GENERALIZED BERNSTEIN PROBLEM: (proposed by SaEarp & Toubiana in 1999)

Are planes the only EW complete multigraphs?

REMARK 1 Bernstein Problem is trivial for the non-minimal type

(but not the Generalized Bernstein Problem)



AIM OF THE TALK (AND SOME REMARKS)

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Are planes the only EW complete multigraphs?

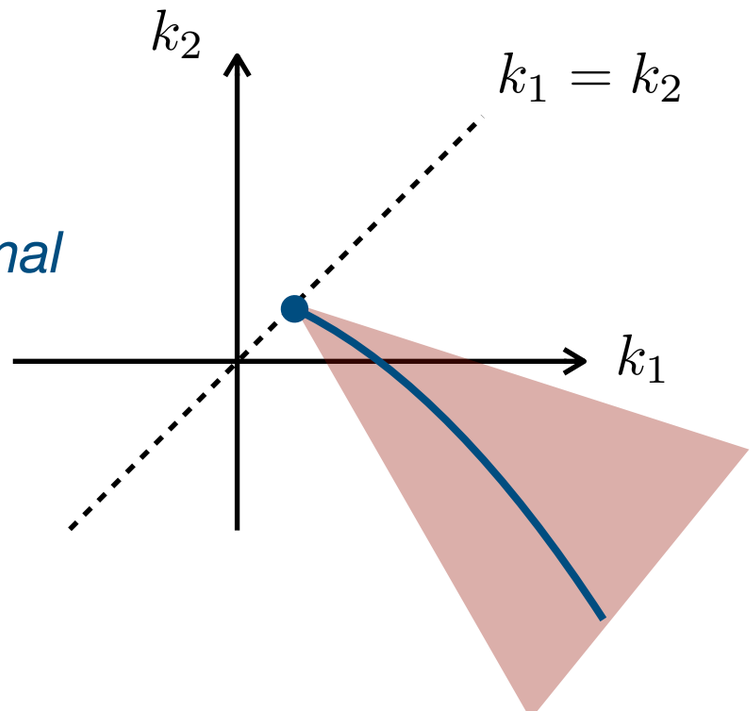
REMARK 1 Bernstein Problem is trivial for the non-minimal type

REMARK 2 Bernstein Problem is solved for uniformly EW surfaces:

(but not the Generalized Bernstein Problem)

PROOF

- We can assume the graph is of minimal type.
- Uniformly EW surfaces of minimal type have *quasiconformal Gauss map,* as noticed by SaEarp&Toubiana, 1999.
- Entire graphs with quasiconformal Gauss map are planes (Simon, 1977)



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Aim of the Talk (and some Remarks)

EW = Elliptic Weingarten:

$$k_2 = f(k_1) \text{ with } f' < 0$$

Bernstein Problem: (proposed by Rosenberg & SaEarp in 1994)

Are planes the only EW entire graphs?

Generalized Bernstein Problem: (proposed by SaEarp & Toubiana in 1999)

Are planes the only EW complete multigraphs?

Remark 1 Bernstein Problem is trivial for the non-minimal type

Remark 2 Bernstein Problem is solved for uniformly EW surfaces

Theorem A: Generalized Bernstein Problem for Uniformly EW surfaces (—,Gálvez,Mira, 2020)

Planes are the only uniformly EW complete multigraphs

AIM OF THE TALK (AND SOME REMARKS)

EW = Elliptic Weingarten:

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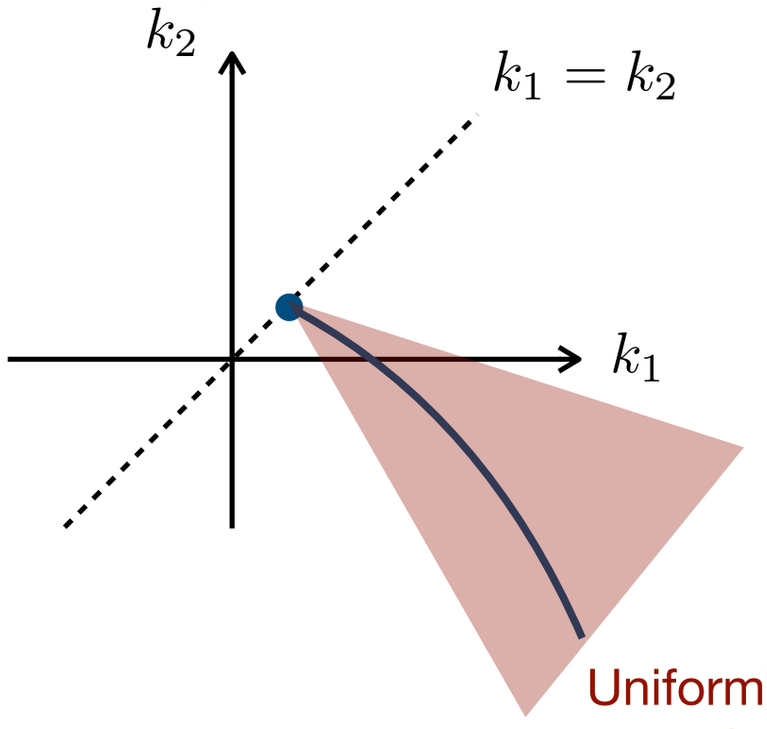
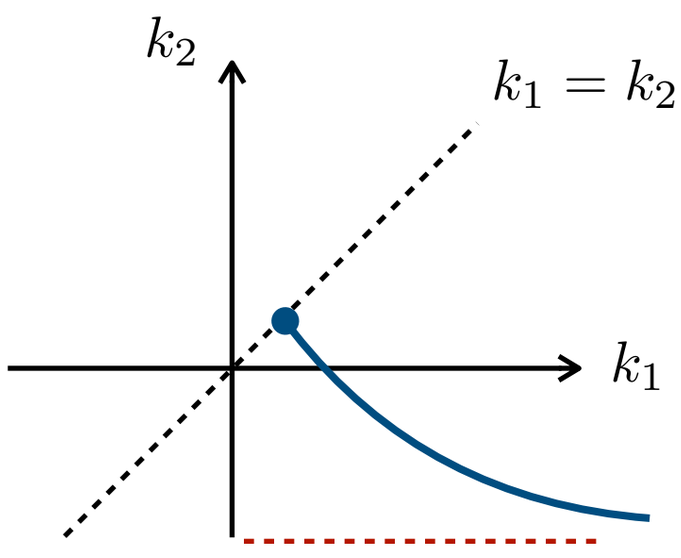
GENERALIZED BERNSTEIN PROBLEM: **(PROPOSED BY SAEARP & TOUBIANA IN 1999)**

Are planes the only EW complete multigraphs?

REMARK 3 Up to reverse the orientation, there are **TWO TYPES** of EW equations:



Uniformly EW are here!



Up to reverse the orientation, there are only

TWO POSSIBILITIES:

$$f(\infty) = -\infty \quad \text{or} \quad f(\infty) \neq -\infty$$

AIM OF THE TALK (AND SOME REMARKS)

EW = Elliptic Weingarten:

$$k_2 = f(k_1) \text{ with } f' < 0$$

BERNSTEIN PROBLEM: **(PROPOSED BY ROSENBERG & SAEARP IN 1994)**

Are planes the only EW entire graphs?

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REMARK 3 Up to reverse the orientation, there are **TWO TYPES** of EW equations:

$$f(\infty) = -\infty \quad \text{or} \quad f(\infty) \neq -\infty$$

THEOREM B: BERNSTEIN THEOREM FOR TYPE II EW-SURFACES (—, GÁLVEZ, MIRA, 2020)

If $f(\infty) \neq -\infty$, planes are the only EW entire graphs

Aim of the Talk

EW = Elliptic Weingarten:

$k_2 = f(k_1)$ with $f' < 0$

Bernstein problem: **(proposed by Rosenberg & Saearp in 1994)**

Are planes the only EW entire graphs?

Generalized Bernstein problem: **(proposed by Saearp & Toubiana in 1999)**

Are planes the only EW complete multigraphs?

Theorem A: Generalized Bernstein Problem for Uniformly EW surfaces (—, Gálvez, Mira, 2020)

Planes are the only uniformly EW complete multigraphs

Theorem B: Bernstein Theorem for Type II EW-Surfaces (—, Gálvez, Mira, 2020)

If $f(\infty) \neq -\infty$, planes are the only EW entire graphs

I. Fernández, J.A. Gálvez, P. Mira,
Quasiconformal Gauss maps and the Bernstein problem for Weingarten multigraphs, Amer. J. Math.

Theorem A: Planes are the only uniformly EW complete multigraphs

Strategy to prove Theorem A

$$||II||^2 = k_1^2 + k_2^2$$

A.1 Uniformly EW multigraphs have bounded 2nd f.f.

- Strategy for CMC surfaces:
  - ROSENBERG-SOUAM-TOUBIANA: Blow-up process (the limit surface is minimal)
  - OSSERMAN: The image of the Gauss map of a complete minimal surface is dense (unless it is a plane)
- Key difficulty here: the limit could be no longer a W-surface!
- Bonus: the limit has quasiconformal Gauss map.

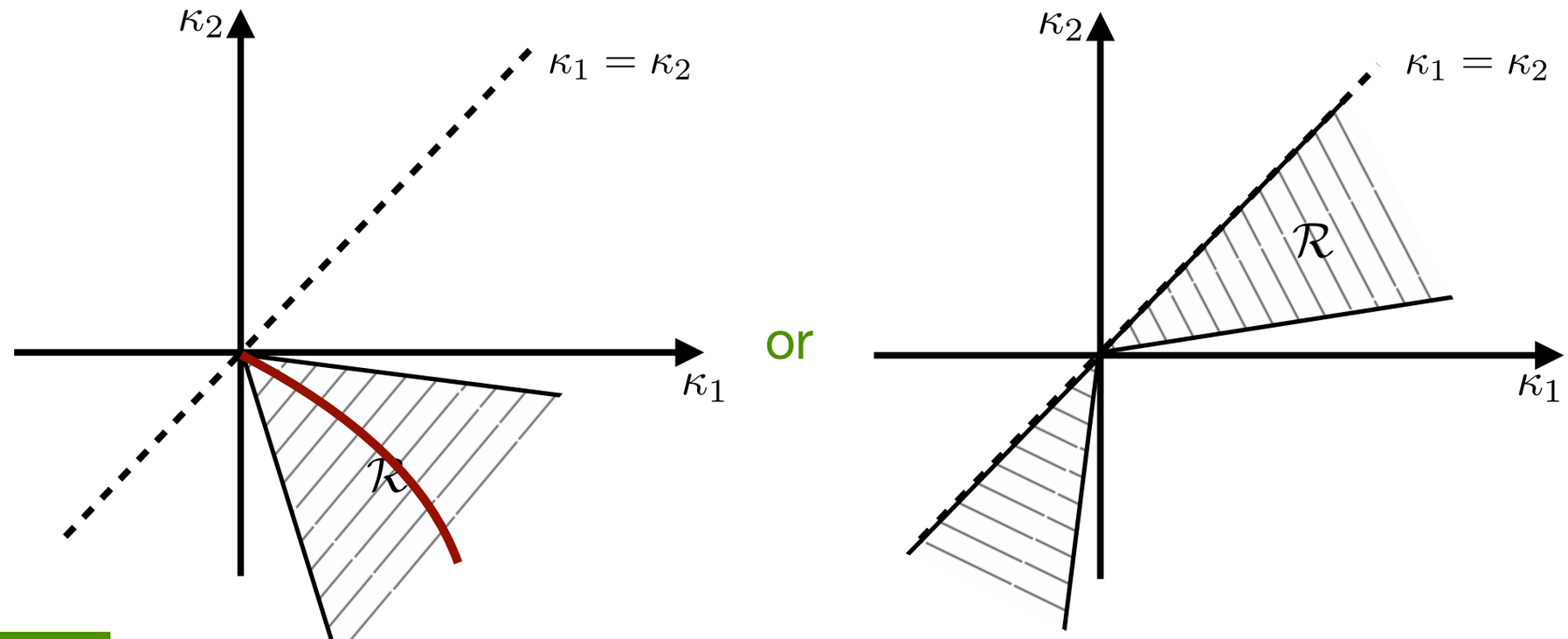
A.2 Planes are the only EW complete multigraphs with bounded 2nd f.f.

Quasiconformal Gauss maps

A surface has quasiconformal Gauss map if its principle curvatures satisfies:

$$\kappa_1^2 + \kappa_2^2 \leq 2\gamma\, \kappa_1\, \kappa_2 \qquad \text{for some } \gamma \in \mathbb{R}$$

That is, if its curvature diagram is contained in a region of the type:



Key facts

- Uniformly EW of minimal type have quasiconformal Gauss map
- EW surfaces of minimal type and bounded 2nd f.f. have quasiconformal Gauss map
- **Simon** (1977): entire graphs with quasiconformal Gauss map are planes

Theorem A: Planes are the only uniformly EW complete multigraphs

A.1 Uniformly EW multigraphs have bounded 2nd f.f.

- Assume S has unbounded 2nd f.f: $\lambda_n := ||II_S(p_n)|| \to \infty$
- Perform a blow-up process

The blow-up:

We can construct a sequence of surfaces $S_n = \lambda_n(D_n - p_n)$

1. $||II_{S_n}|| < 2$ and $||II_{S_n}(0)|| = 1$,
2. $diam(S_n) \to \infty$
3. S_n are uniformly elliptic Weingarten surfaces, with the same bounds for the uniformly ellipticity condition.

The convergency:

1. S_n are uniformly bounded in the C^2 norm.
2. Thanks to the uniformly elliptic condition, they are also uniformly bounded in the $C^{2,\alpha}$ norm (Nirenberg).
3. Thus, they converge in some $C^{2,\beta}$ norm to a surface S_0.

The limit surface may **not be EW**, but:

1. is complete, non-planar, with bounded 2nd f.f.
2. has quasiconformal Gauss map
3. is a multigraph (the Gauss map is open)

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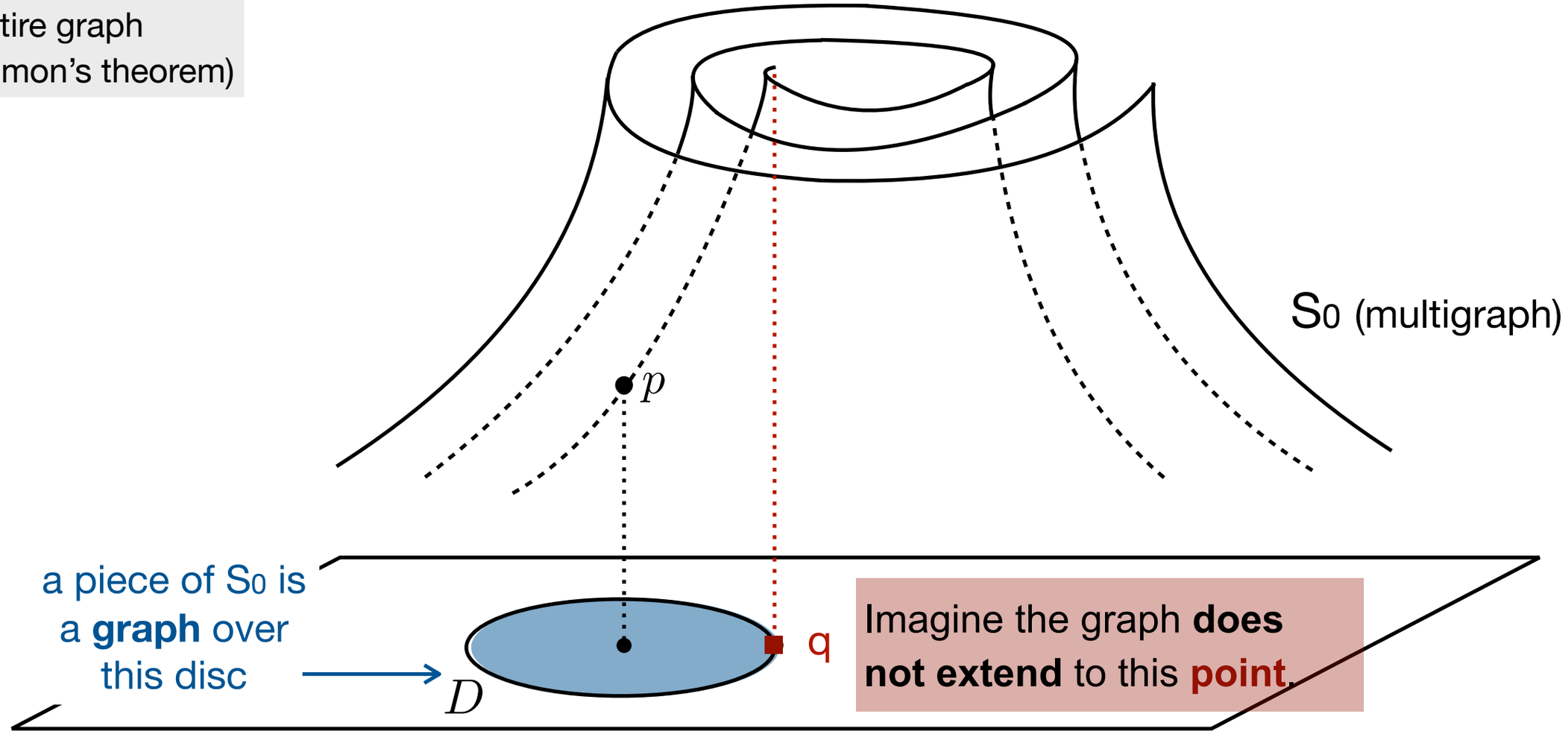
- Perform a blow-up process
- Obtain a non-planar complete multigraph S_0 with bounded 2nd.f.f. and quasiconformal Gauss map
- **GET A CONTRADICTION:**

It cannot be an entire graph (Simon's theorem)



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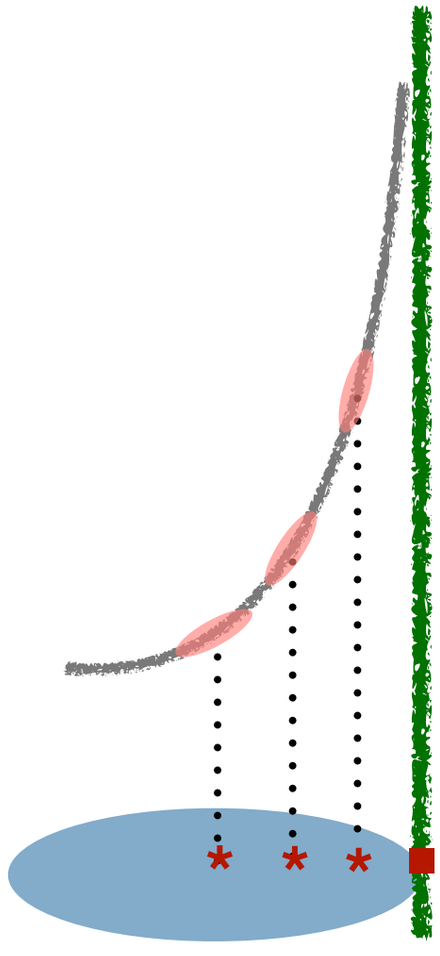
Bounded 2 f.f. implies that

$\exists \delta > 0$ s.t. $D_\delta(p) \subset S_0$ has uniformly bounded geometry $\forall p \in S$

a piece of S_0 is a **graph** over this disc →



- the tangent planes become vertical.
- the discs converge $\mathcal{C}^{1,\alpha}$ to a surface.
- the limit surface has quasiconformal Gauss map N.
- the image of N is contained in a closed hemisphere.
- N must be constant (non-constant quasiconformal maps are open)

Thus, the graph is asymptotic to a vertical plane **[HRS]**

Imagine the graph **does not extend** to this **point**.

Theorem A: Planes are the only uniformly EW complete multigraphs

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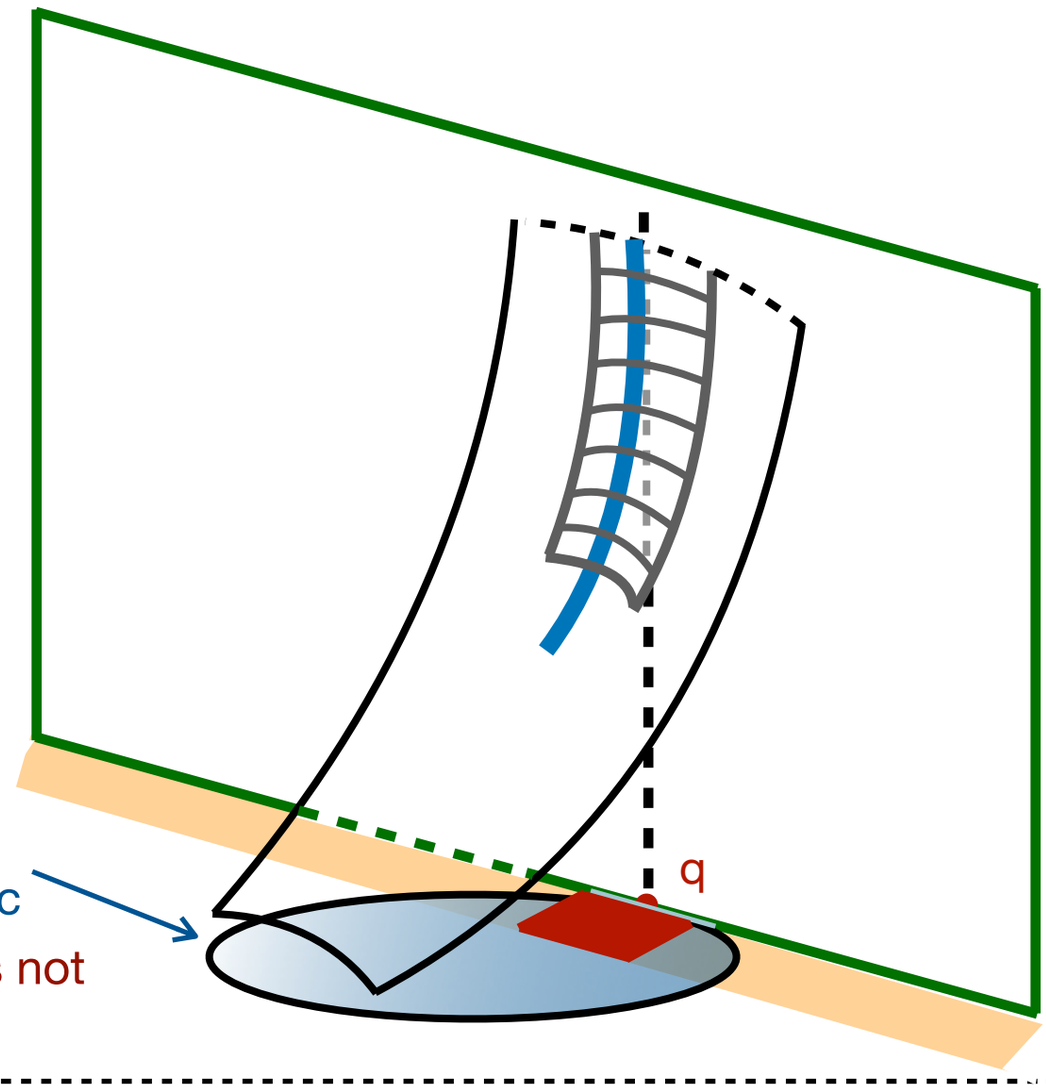

A piece of S_0 is a **graph** over this disc and the graph does not extend to q

The graph is asymptotic to a (piece of) **vertical plane**

The graph extends to a semi-tubular neighborhood (and it is asymptotic to the plane)

We can **iterate** this process and extend the **graph** along the **line**

Theorem A: Planes are the only uniformly EW complete multigraphs

A.1 Uniformly EW multigraphs have bounded 2nd f.f.

- Assume S has unbounded 2nd f.f: $\lambda_n := ||II_S(p_n)|| \to \infty$
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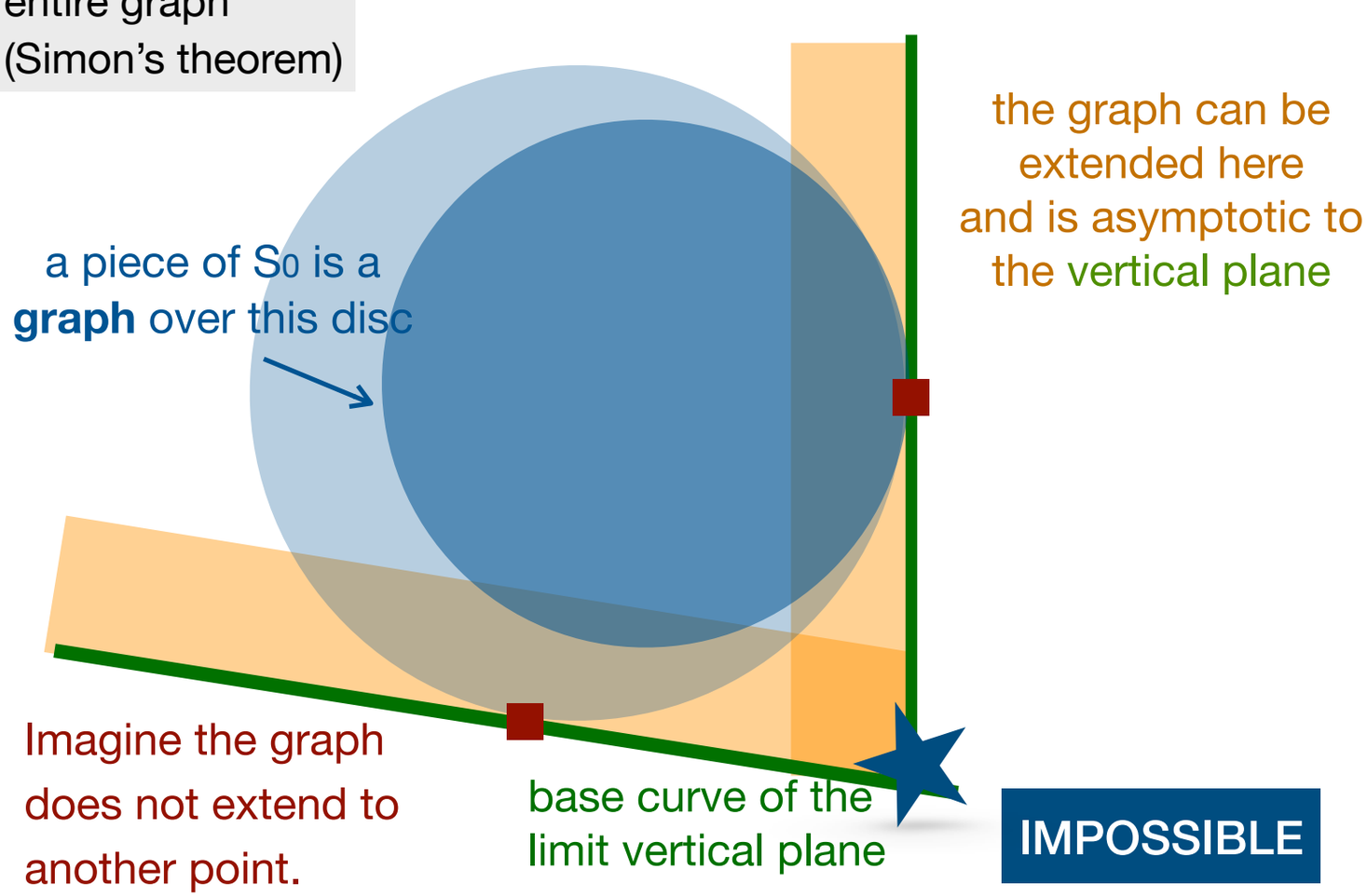

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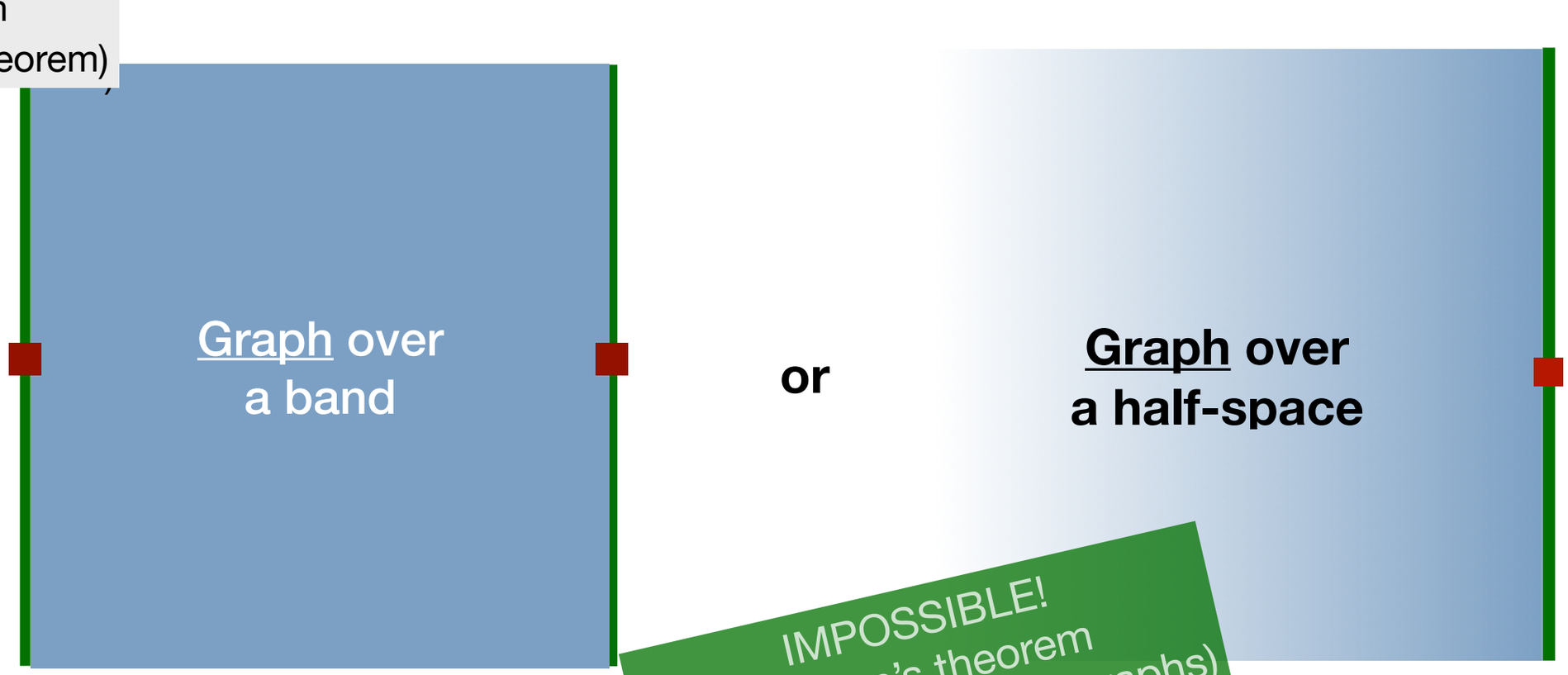

Theorem A: Planes are the only uniformly EW complete multigraphs

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- Assume S has unbounded 2nd f.f: $\lambda_n := ||II_S(p_n)|| \to \infty$
- Perform a blow-up process
- Obtain a non-planar complete multigraph S_0 with bounded 2nd.f.f. and quasiconformal Gauss map
- **IMPOSSIBLE!**

QED

Theorem A: Planes are the only uniformly EW complete multigraphs

A.1 Uniformly EW multigraphs have bounded 2nd f.f.

And also:

Planes are the only complete multigraphs with bounded 2nd f.f. and quasiconformal Gauss map.

A.2 EW complete multigraphs with bounded 2nd f.f. are planes

- If the surface is of minimal type, then it has quasiconformal Gauss map and the result is true by the previous step.
- If the surface is of non-minimal type…

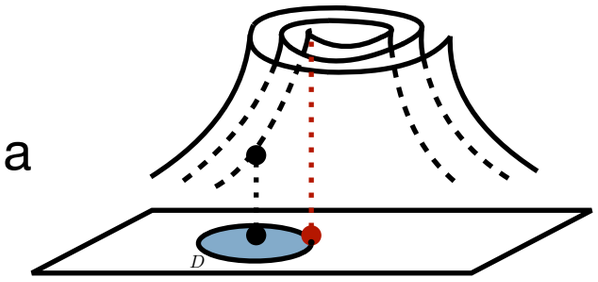
Theorem A: Planes are the only uniformly EW complete multigraphs

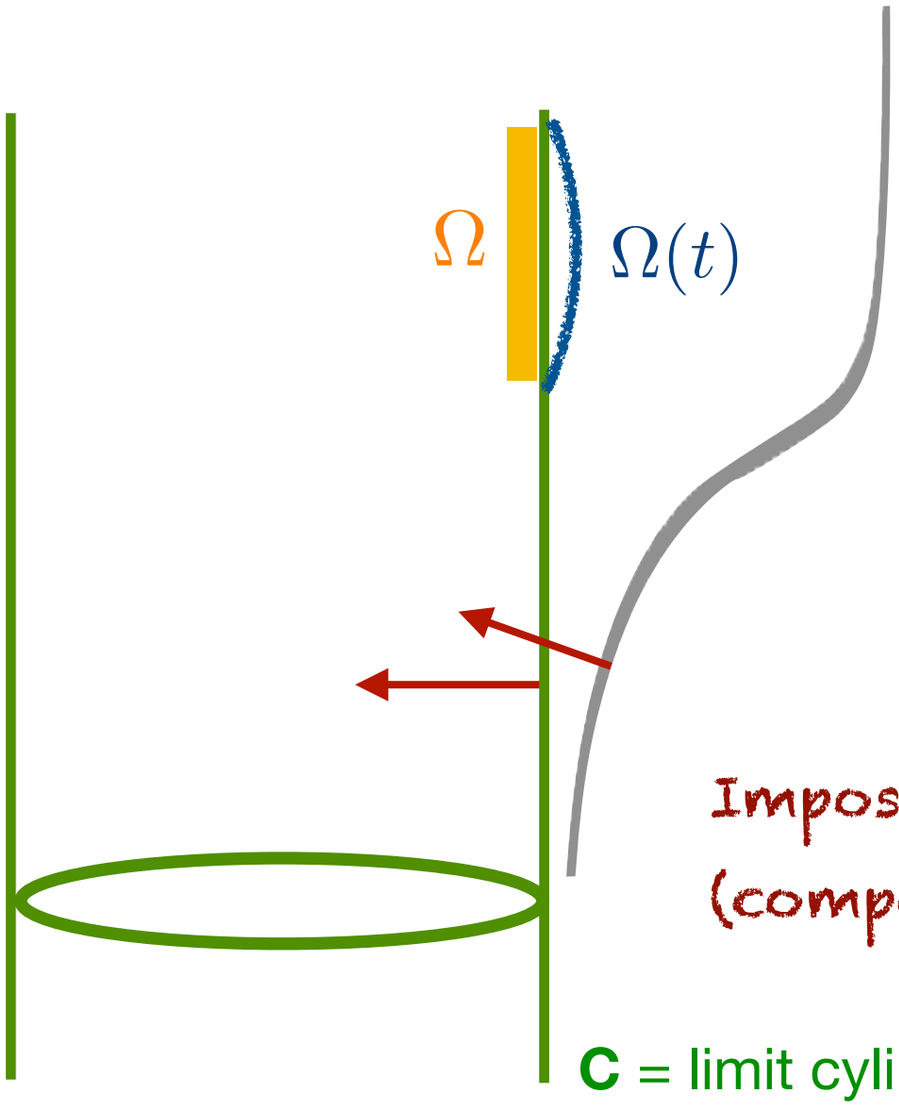
A.2 EW complete multigraphs with bounded 2nd f.f. are planes

- If the surface is of non-minimal type…

Similar arguments as in the previous step lead to:

a piece of S can be extended to a graph that is asymptotic to a piece of a vertical cylinder





$\exists\, \Omega \subset C$ relatively compact domain

$\exists\, \Omega_t$ deformation with $\partial\Omega_t = \partial\Omega$

$W(k_1(t), k_2(t)) < 0$

TRANSLATE DOWNWARDS

Impossible!

(comparison principle for surfaces)

C = limit cylinder

Q.E.D

Theorem A: Planes are the only uniformly EW complete multigraphs

Summary of the proof

A.1 Prove *curvature estimates* (bounds for the 2nd f.f.) for uniformly EW multigraphs.

Assume S has unbounded 2nd f.f.

- blow-up process converging to a non-planar multigraph with bounded 2nd f.f. and quasiconformal Gauss map.
- planes are the only complete multigraphs with bounded 2nd f.f. and quasiconformal Gauss map.

Thus, S has bounded 2nd f.f.

A.2 Planes are the only EW complete multigraphs with bounded 2nd f.f.

- Minimal type: the Gauss map is quasiconformal. (A.2. is true thanks to the previous result)
- Non-minimal type: control of the asymptotic behavior + deformation arguments

Q.E.D.

Theorem B: Planes are the only EW entire graphs if $f(\infty) \neq -\infty$

PROOF

Since *f* is bounded, we can find a such that the parallel surface at distance a is:

- a (regular) complete multigraph
- with bounded 2nd f.f. and
- quasiconformal Gauss map.

Thus, the result follows from the fact that

Planes are the only complete multigraphs with bounded 2nd f.f. and quasiconformal Gauss map.

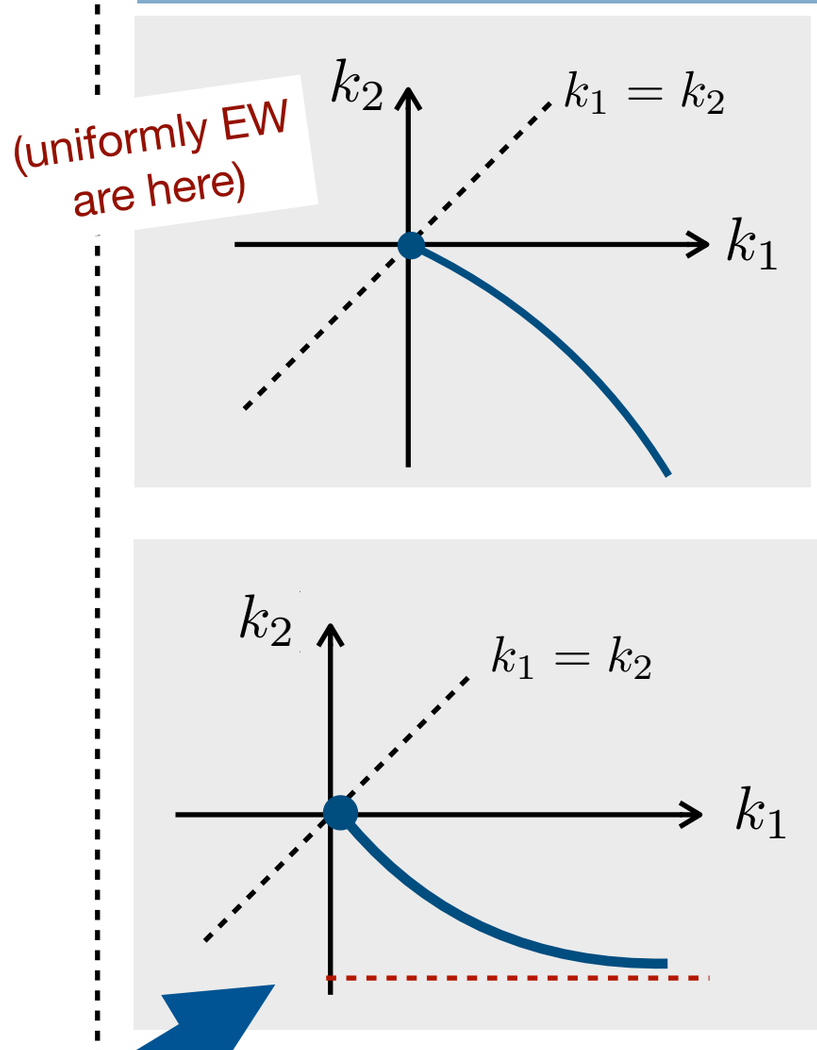
(ALREADY PROVED IN THEOREM A)

Q.E.D.

EW surface: $k_2 = f(k_1)\,, f' < 0$

TWO POSSIBILITIES FOR *f* :

$f(\infty) = -\infty$ or $f(\infty) \neq -\infty$



And That's The End...

EW = Elliptic Weingarten:

$$k_2 = f(k_1) \text{ with } f' < 0$$

BERNSTEIN PROBLEM: **(PROPOSED BY ROSENBERG & SAEARP IN 1994)**

Are planes the only EW entire graphs?

GENERALIZED BERNSTEIN PROBLEM: **(PROPOSED BY SAEARP & TOUBIANA IN 1999)**

Are planes the only EW complete multigraphs?

RESULTS

- **Theorem A:** Planes are the only uniformly EW complete multigraphs
- **Theorem B:** Planes are the only EW entire graphs if $f(\infty) \neq -\infty$

I. Fernández, J.A. Gálvez, P. Mira,
Quasiconformal Gauss maps and the Bernstein problem for Weingarten multigraphs, Amer. J. Math.

FURTHER QUESTIONS

- The **Bernstein problem** remains **open** when f is not bounded and not uniformly elliptic
- Related with **Simon's theorem**: are planes the only complete multigraphs with quasiconformal Gauss map?

THE BERNSTEIN PROBLEM FOR WEINGARTEN SURFACES

¡GRACIAS!

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I. Fernández, J.A. Gálvez, P. Mira:

Quasiconformal Gauss maps and the Bernstein problem for Weingarten multigraphs,

to appear in American Journal of Mathematics.

Available at ArXiv