A Twin Correspondence for Surfaces in Killing submersions

$\mathcal{H}$-Workshop 2023

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Based on:

A. Del Prete, J.M. Manzano
A duality for prescribed mean curvature graphs in Riemannian and Lorentzian Killig submersions.
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Contents

Definition

A ***Killing submersion*** is a (semi-)Riemannian submersion $\pi\colon \mathbb{E} \to M$, where $\mathbb{E}$ and M are orientable and connected, such that the fibers of π are the integral curves of a nowhere vanishing (temporal) Killing vector field ξ.

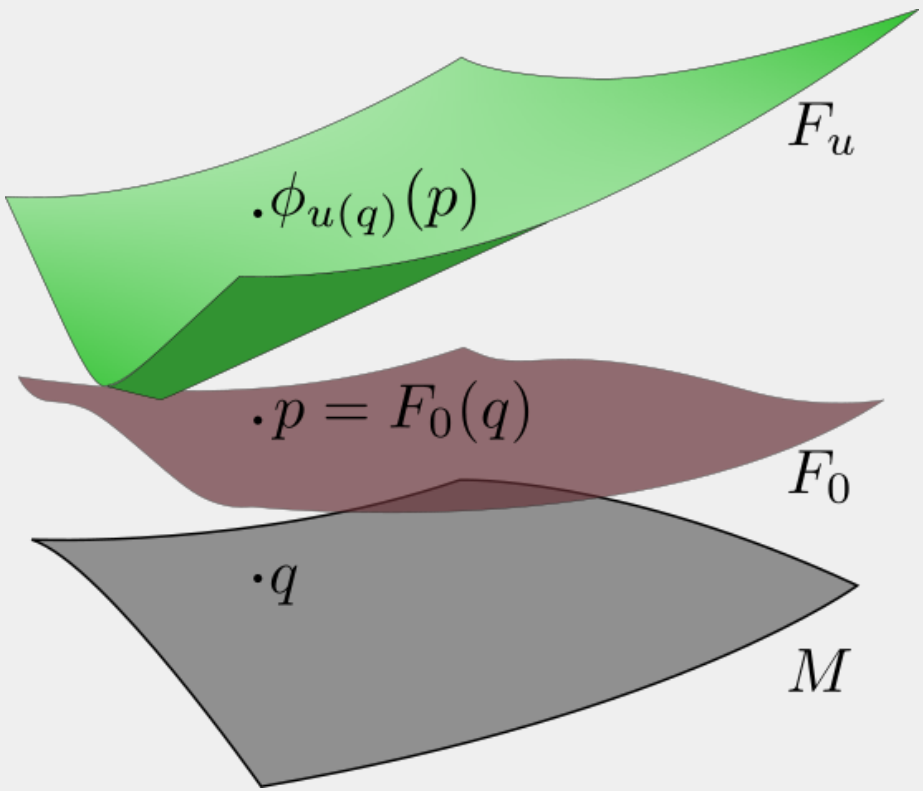
$G = \{\phi_t\}_t$: one-parameter group of isometries of $\mathbb{E}$ associated to ξ (**vertical translations**).
If $G \curvearrowright \mathbb{E}$ is free and proper $\implies$ $\pi\colon \mathbb{E} \to M$ is a *Killing submersion*, where $M = \mathbb{E}/G$ is a unique orientable Riemannian surface.

Basic Features:

- **Killing length**: $\mu = \|\xi\| \in \mathcal{C}^\infty(\mathbb{E})$.
- **Bundle curvature**: $\tau \in \mathcal{C}^\infty(\mathbb{E})$ such that $\tau(p) = -\frac{\varepsilon}{\mu(p)}\langle \overline{\nabla}_{e_1}\xi, e_2\rangle$
- **Theorem (Lerma–Manzano):** If $\mathbb{E}$ is simply connected $\implies$ $\mathbb{E} = \mathbb{E}(M, \mu, \tau, \varepsilon = \pm 1)$
- If a fiber has finite length, then all have finite length.
- π admits a global section provided that:
  - the base M is not compact, or
  - the fibers of π have infinite length $\left(\int_M \frac{\tau}{\mu} \neq 0\right)$.

$$\mathbb{E}(M, \tau, \mu) = \mathbb{E}(M, \tau, \mu, 1) \qquad \mathbb{L}(M, \tau, \mu) = \mathbb{E}(M, \tau, \mu, -1)$$

1) Killing submersions: Killing graphs



$$d \in \mathcal{C}^\infty(\mathbb{E}) \text{ is s.t. } p = \phi_{d(p)}(F_0(\pi(p)))$$

$$u \in \mathcal{C}^2(M) \implies \bar{u} \in \mathcal{C}^2(\mathbb{E}) \implies (d - \bar{u}) \in \mathcal{C}^2(\mathbb{E})$$

$$N = \frac{\varepsilon \overline{\nabla}(d - \bar{u})}{\left\| \overline{\nabla}(d - \bar{u}) \right\|_{\mathbb{E}}}$$

$$\|\overline{\nabla}(d - \overline{u})\|^2_{\mathbb{E}} = \tfrac{\epsilon}{\mu^2} + \|Gu\|^2$$

where $Gu = \nabla u - Z$ with $Z = \pi_*(\overline{\nabla} d) \in \mathfrak{X}(M)$

$$2H\mu = \operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1 + \varepsilon\mu^2 \|Gu\|^2}} \right)$$

1) Killing submersions: local model

$$(M, g_M) \simeq (U \subset \mathbb{R}^2, \lambda_1^2 dx^2 + \lambda_2^2 dy^2)$$
$$e_1 = \frac{\partial x}{\lambda_1},\ e_2 = \frac{\partial y}{\lambda_2} \text{ orthonormal basis}$$

$$\mathbf{C}_{M,\tau,\mu}(x,y) = 2\int_0^1 s\,\frac{\tau(sx,sy)\,\lambda_1(sx,sy)\,\lambda_2(sx,sy)}{\mu(sx,sy)}\mathrm{d}s.$$

$$a = \frac{-\epsilon y\,\mathbf{C}_{M,\tau,\mu}}{\lambda_1}, \qquad b = \frac{\epsilon x\,\mathbf{C}_{M,\tau,\mu}}{\lambda_2}.$$

$$\Longrightarrow (\mathbb{E}, g_{\mathbb{E}}) \simeq (U \times \mathbb{R}, ds^2)$$
$$ds^2 = \lambda_1^2 dx^2 + \lambda_2^2 dy^2 + \varepsilon\mu^2\,(dz - \lambda_1 a dx - \lambda_2 b dy)^2$$

$$E_1 = \frac{\partial x}{\lambda_1} + a\partial z \quad E_2 = \frac{\partial y}{\lambda_2} + b\partial z \quad E_3 = \frac{\partial z}{\mu}$$

$$Z = ae_1 + be_2 \implies \operatorname{div}(JZ) = \frac{-2\varepsilon\tau}{\mu}$$

$$\begin{aligned}\overline{R}(X,Y,Z,W) = &-\tau^2\langle X\times Y, Z\times W\rangle - (K_M - 4\epsilon\tau^2)\langle X\times Y, E_3\rangle\langle Z\times W, E_3\rangle\\ &+\langle X\times Y, E_3\rangle\langle Z\times W, E_3\times T\rangle + \langle Z\times W, E_3\rangle\langle X\times Y, E_3\times T\rangle\\ &+\tfrac{\epsilon}{\mu}\overline{\mathrm{Hess}}(\mu)((X\times Y)\times E_3, (Z\times W)\times E_3)\end{aligned}$$

where $T = \overline{\nabla}\tau + \frac{2\tau}{\mu}\overline{\nabla}\mu$

Theorem: Conformal duality

Let M be a simply connnected Riemannian surface and let $\tau, H, \mu \in \mathcal{C}^\infty(M)$ be arbitrary functions such that $\mu > 0$. There is a bijective correspondence between

a) entire graphs in $\mathbb{E}(M, \tau, \mu)$ with prescribed mean curvature H, and

b) entire spacelike graphs in $\mathbb{L}(M, H, \mu^{-1})$ with prescribed mean curvature τ.

Sketch of the proof:

$$F_0 \colon M \to \mathbb{E}(M, \tau, \mu) \qquad\qquad \tilde{F}_0 \colon M \to \mathbb{L}(M, H, 1/\mu)$$

$$\implies \exists Z \in \mathfrak{X}(M) \mid \operatorname{div}(JZ) = -2\tau/\mu \qquad \implies \exists \tilde{Z} \in \mathfrak{X}(M) \mid \operatorname{div}(J\tilde{Z}) = 2H\mu$$

$$u \in \mathcal{C}^\infty(M) \mid \operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}}\right) = 2H\mu = \operatorname{div}(J\tilde{Z})$$

$$\implies \exists v \in \mathcal{C}^\infty(M) \mid \frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}} - J\tilde{Z} = -J\nabla v \implies \frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}} = -J\tilde{G}v$$

$$\frac{\mu^4\|Gu\|^2}{1+\mu^2\|Gu\|^2} = \|\widetilde{G}v\|^2 \Leftrightarrow \frac{1}{1+\mu^2\|Gu\|^2} = 1 - \mu^{-2}\|\widetilde{G}v\|^2.$$

$$\operatorname{div}\left(\frac{\mu^{-2}\widetilde{G}v}{\sqrt{1+\mu^{-2}\|\widetilde{G}v\|^2}}\right) = \operatorname{div}(JGu) = \operatorname{div}(J\nabla u) - \operatorname{div}(JZ) = \frac{2\tau}{\mu},$$

Theorem

If $H \in \mathcal{C}^\infty(\mathbb{R}^2)$ is such that

1. *H is invariant with respect to a one-parameter group of isometries of $\mathbb{R}^2$, or*
2. *H is bounded and ∇H is also bounded,*

then, there exists an entire spacelike graph in $\mathbb{L}^3$ with mean curvature H.

Sketch of the proof:

Idea: Find an entire minimal graph in $\mathbb{R}^3_H = \mathbb{E}(\mathbb{R}^2, H, 1)$ and use the conformal duality.

H invariant by isometry $\implies$ $H = H\left(\sqrt{x^2+y^2}\right)$ or $H = H(x)$.

$$H = H\left(\sqrt{x^2+y^2}\right)$$
$$\mathbb{R}^3_H \equiv (\mathbb{R}^3, ds^2)$$
$$ds^2 = dx^2 + dy^2 + [dz + \mathbf{C}(ydx - xdy)]^2$$
$z = 0$ is minimal.

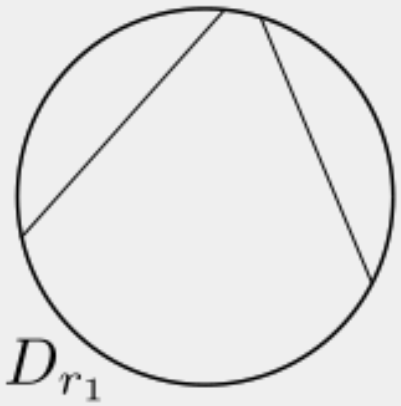
$$H = H(x)$$
$$\mathbb{R}^3_H \equiv (\mathbb{R}^3, ds^2)$$
$$ds^2 = dx^2 + dy^2 + \left[dz - \left(\int 2H(x)dx\right) dy\right]^2$$
$z = 0$ is minimal.

Lemma

For each $r > 0$, there is a minimal graph Σ_r in $\mathbb{R}^3_H$ over $D_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$ with vertical normal at $(0,0)$.

3) Entire spacelike graphs with prescribed mean curvature in $\mathbb{L}^3$

$\forall n \in \mathbb{N}$, Σ_n is the minimal graph of u_n in D_n with $Gu(0,0) = (0,0)$ and $u(0,0) = 0$
$\forall r > 0$, $D_r \times \mathbb{R} \subset \mathbb{R}^3_H$ has bounded geometry $\implies$ $\exists \Sigma_{r,\infty} = \lim\limits_{n\to\infty} \Sigma_n$ in D_r



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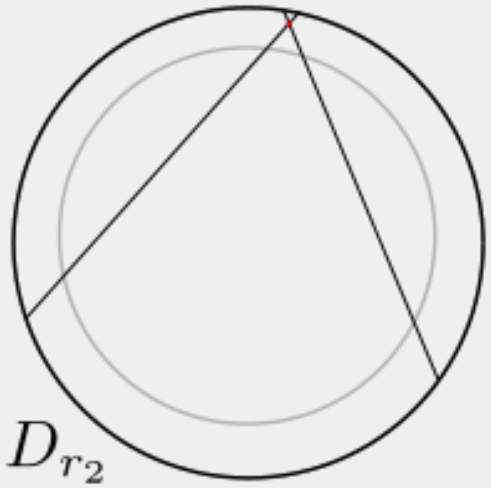

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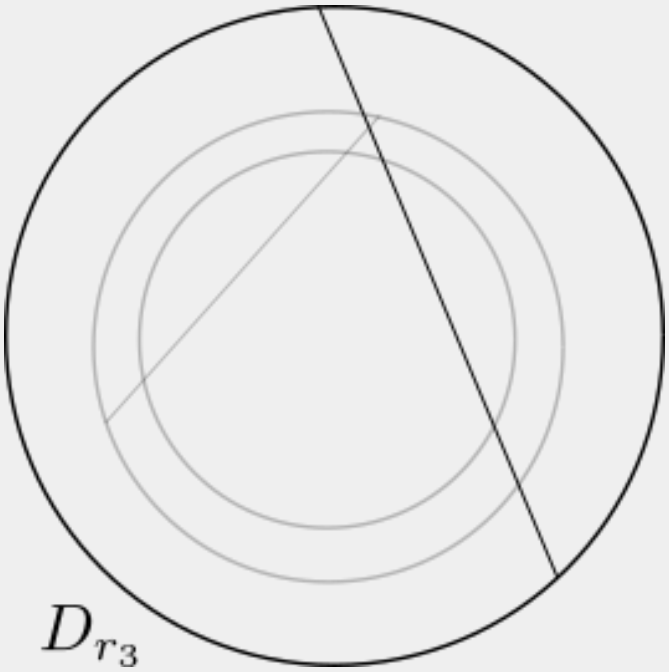

$\Sigma_{r_n,\infty} \subset \Sigma_{r_{n+1},\infty}$ $\implies$ $\exists \Sigma_\infty = \lim_{n\to\infty} \Sigma_{r_n,\infty}$ will be a complete minimal graph over a strip, or a halfplane, or the entire plane.
H and ∇H bounded $\implies$ Mazet's general halfspace theorem can be applied to discard the first two cases.

Thanks for your attention!