On constant mean curvature surfaces in product manifolds

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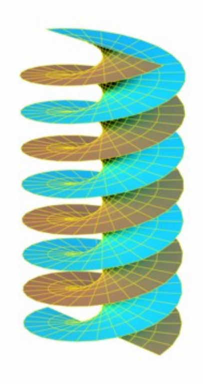
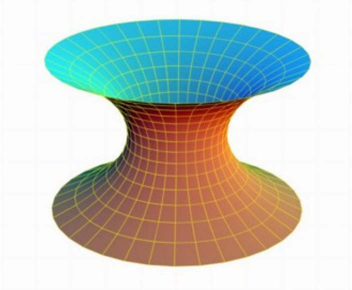
Helicoid and catenoid



helicoid



catenoid

(images : Matthias Weber)

The local problem in space forms

- $\mathbb{M}^3(c)$ = space form for constant sectional curvature c ($\mathbb{S}^3(c)$ for $c > 0$, $\mathbb{R}^3$ for $c = 0$, $\mathbb{H}^3(c)$ for $c < 0$).
- Σ a simply connected Riemannian surface.
- $\mathrm{d}s^2$ its metric.
- ∇ its Levi-Civita connection.
- K the curvature of $\mathrm{d}s^2$.
- Given a field $S : \mathrm{T}\Sigma \to \mathrm{T}\Sigma$ of symmetric operators, then there exists an isometric immersion from Σ into $\mathbb{M}^3(c)$ with S as induced shape operator if and only if $(\mathrm{d}s^2, S)$ satisfies the Gauss and Codazzi equations

$$K = \det S + c, \tag{1}$$

$$\forall (X, Y) \in \mathcal{X}(\Sigma)^2, \nabla_X SY - \nabla_Y SX - S[X, Y] = 0. \tag{2}$$

The associate family

- Let $H \in \mathbb{R}$.
- J is the $\pi/2$-rotation on $\mathrm{T}\Sigma$.
- $x : \Sigma \to \mathbb{M}^3(c)$ constant mean curvature (CMC) H isometric immersion.
- Then x belongs to a one-paramater family $(x_\theta)_{\theta \in \mathbb{R}/2\pi\mathbb{Z}}$ such that, for each θ, $x_\theta : \Sigma \to \mathbb{M}^3(c)$ is a CMC H isometric immersion. This family is called the associate family of x.
- This family is non constant in θ unless x is totally umbilical.
- Indeed, if x has $(\mathrm{d}s^2, S)$ satisfying Gauss-Codazzi with S symmetric and $\operatorname{tr} S = 2H$, then, for any θ, $(\mathrm{d}s^2, S_\theta)$ satisfies Gauss-Codazzi for $S_\theta = e^{\theta J}(S - H\mathrm{I}) + H\mathrm{I}$, with S_θ symmetric and $\operatorname{tr} S_\theta = 2H$.
- For $\theta = \pm\pi/2$, the immersion is said to be the conjugate of x.

Rigidity

Theorem (Calabi, Lawson)

Two CMC H isometric immersions from Σ into $\mathbb{M}^3(c)$ are associate.

Existence

- Necessary condition (by Gauss):

$$K \leqslant c + H^2.$$

- "Almost necessary and sufficient" condition: the Ricci condition and its generalization.

Theorem (Ricci, Lawson, do Carmo - Dajczer)

Assume that $K < c + H^2$. Then there exists a CMC H isometric immersion $\Sigma \to \mathbb{M}^3(c)$ if and only if the metric $(-K + c + H^2)\mathrm{d}s^2$ has curvature $1 + \frac{c+H^2}{K-c-H^2}$.

This is particularly interesting when $c + H^2 = 0$ (e.g., minimal surfaces in $\mathbb{R}^3$), since, in that case, the curvature of the metric $(-K + c + H^2)\mathrm{d}s^2$ is constant.

Existence

Theorem (A. Moroianu - S. Moroianu)

There exists a minimal isometric immersion from Σ *into* $\mathbb{R}^3$ *if and only if* $K \leqslant 0$ *and*

$$||\nabla K||^2 = K\Delta K - 4K^3. \tag{3}$$

Note that, on an open set on which $K < 0$, (3) is equivalent to the Ricci condition.

The global problem in $\mathbb{R}^3$

Theorem (H. I. Choi - Meeks - White, 1990)

Any properly embedded minimal surface with more than one end admits a unique isometric minimal immersion into $\mathbb{R}^3$.

Conjecture (Meeks)

Except for the helicoid, the inclusion map of a complete embedded constant mean curvature surface is the unique such isometric immersion with the same constant mean curvature.

Theorem (Meeks - Tinaglia, 2011)

The conjecture is true in the following cases:

- *if moreover the surface is minimal and has finite genus,*
- *or if moreover the surface is non-minimal, has finite genus and bounded Gaussian curvature.*

The local problem in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

- ξ unit vertical field, i.e., tangent to the factor $\mathbb{R}$.
- If $x : \Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$ is an isometric immersion with normal N, then we write $\xi = \mathrm{d}x(T) + \nu N$ with $T \in \mathcal{X}(\Sigma)$ and $\nu : \Sigma \to [-1, 1]$ (the angle function).
- $(\mathrm{d}s^2, S, T, \nu)$ are called Gauss-Codazzi data of x.
- A necessary and sufficient condition for the existence of an isometric immersion $x : \Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$ with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$ is that $(\mathrm{d}s^2, S, T, \nu)$ satisfy the following set of equations:

$$K = \det S + c\nu^2, \tag{C1}$$

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = c\nu(\langle Y, T\rangle X - \langle X, T\rangle Y), \tag{C2}$$

$$\nabla_X T = \nu SX, \tag{C3}$$

$$\mathrm{d}\nu(X) + \langle SX, T\rangle = 0. \tag{C4}$$

$$||T||^2 + \nu^2 = 1. \tag{C5}$$

- Moreover, the immersion is then unique up to congruences preserving the orientations of $\mathbb{M}^2(c)$ and $\mathbb{R}$.
- (C1) and (C2) are the Gauss and Codazzi equations.

The associate family for minimal isometric immersions

- Let $x : \Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$ be a minimal isometric immersion with Gauss-Codazzi data $(\mathrm{d}s^2, S, T, \nu)$.
- Then, for any $\theta \in \mathbb{R}$, $(\mathrm{d}s^2, e^{\theta J}S, e^{\theta J}T, \nu)$ satisfies (C1)-(C5) and $e^{\theta J}S$ symmetric and traceless.
- So there exists a minimal isometric immersion $x_\theta : \Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$ with Gauss-Codazzi data $(\mathrm{d}s^2, e^{\theta J}S, e^{\theta J}T, \nu)$. This was also proved by Hauswirth, Sá Earp and Toubiana using harmonic maps.

Non associate minimal isometric immersions

The analogue of Calabi's theorem is false in $\mathbb{H}^2 \times \mathbb{R}$ $(c < 0)$.

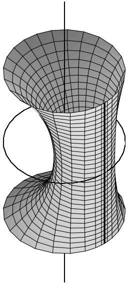


Figure: A minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ foliated by horocycles, with intrinsic curvature -1, and conjugate to a helicoid of pitch 1.

Theorem (Torralbo-Urbano, 2015 (for $c > 0$); D., 2015)

Let Σ be a minimal surface in $\mathbb{M}^2(c) \times \mathbb{R}$ with constant intrinsic curvature K. Then

- *either Σ is totally geodesic and $K = 0$ or $K = c$,*
- *either $c < 0$, $K = c$ and Σ is part of an associate surface of the helicoid of pitch 1.*

Theorem (D., 2015)

Let Σ be a simply connected Riemannian surface with non constant curvature. Then the set of minimal isometric immersions $\Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$ is empty or consists of n families of associate minimal immersions for some integer $n \in [[1, 6]]$.

Corollaire (D., 2015)

Let Σ be a simply connected Riemannian surface with non constant curvature. Let $(f_t)_{t \in I}$ be a continuous family of minimal isometric immersions from Σ to $\mathbb{M}^2(c) \times \mathbb{R}$, where I is a real interval. Then all immersions f_t, $t \in I$, are associate.

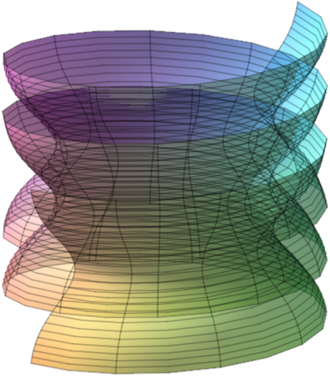
The CMC case

Theorem (D. - Domingos - Vitório)

Let Σ be a non minimal CMC surface in $\mathbb{M}^2(c) \times \mathbb{R}$ with constant intrinsic curvature K. Then Σ is part of

- *either a vertical surface over a curve of non zero constant curvature in $\mathbb{M}^2(c)$ and then $K = 0$,*
- *either an Abresch-Rosenberg-Leite surface when $c < 0$ and then $K = 4H^2 + c$,*
- *a certain Sa Earp-Toubiana helicoidal surface when $c < 0$ (new example) and then $K = 4H^2 + c$.*

The CMC case



a CMC helicoidal surface with constant instrinsic curvature in $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$

Sketch of the proof

The first step is to show that, in the context of CMC H isometric immersions, the system (C1)-(C2)-(C3)-(C4)-(C5) can be reduced to

$$\|\nabla\nu + H\nabla h\|^2 = (H^2 - K + c\nu^2)(1-\nu^2), \tag{4}$$

$$\Delta\nu = (2K - c(1+\nu^2) - 4H^2)\nu, \tag{5}$$

$$\|\nabla h\|^2 = 1-\nu^2, \tag{6}$$

$$\Delta h = 2H\nu, \tag{7}$$

where $h:\Sigma\to\mathbb{R}$ is the height function ($T=\nabla h$).
More precisely, assuming that $\nu^2<1$, then $(\mathrm{d}s^2, S, \nabla h, \nu)$ satisfies (C1)-(C2)-(C3)-(C4)-(C5) with $\operatorname{tr} S = 2H$ if and only if $(\mathrm{d}s^2, \nu, h)$ satisfy (4)-(5)-(6)-(7).

Sketch of the proof

The second step is to apply the Weitzenböck-Bochner formula to $\phi = \nu + Hh$:

$$\frac{1}{2}\Delta\|\nabla\phi\|^2 = \langle\nabla\phi, \nabla\Delta\phi\rangle + \|\nabla^2\phi\|^2 + K\|\nabla\phi\|^2,$$

and to derive a new equation:

$$\begin{aligned}0 = 6Hc(H^2 - K - c\nu^2)\langle\nabla\nu, \nabla h\rangle + 4H^2c\nu^2(H^2 - K - 2c + 3c\nu^2)\\ -4(H^2 - K + c\nu^2)(K - c - H^2)(K + 2c\nu^2).\end{aligned}$$

Sketch of the proof

Next we consider a function $q : \Sigma \to \mathbb{R}_+$ introduced by Espinar and Rosenberg; this function q is a normalization of the squared norm of the holomorphic Abresch-Rosenberg differential.
Espinar and Rosenberg showed the following formulas:

$$\|\nabla\nu\|^2 = \frac{4H^2 + c - c\nu^2}{4c}(4(H^2 - K + c\nu^2) + c(1 - \nu^2)) - \frac{q}{c},$$

$$\Delta \log q = 4K$$

(away from the zeroes of q).

Sketch of the proof

- Using all these equations, we get that, if $K \neq 4H^2 + c$, then ν is a root of a degree 18 polynomial, hence constant.
- If ν is constant, then we conclude by the classification of Espinar-Rosenberg of CMC surfaces with constant ν.
- If ν is not constant, then $K = 4H^2 + c$ and some equations simplify to

$$\|\nabla\nu\|^2 = -\frac{1}{c}(4H^2 + c - c\nu^2)^2, \tag{8}$$

$$\Delta\nu = (4H^2 + c - c\nu^2)\nu. \tag{9}$$

In this case we are able to do explicit computations and to recover the surface from a solution to an ODE.

Parallel mean curvature surface

A parallel mean curvature surface in a Riemannian manifold M^n is a surface whose mean curvature vector $\mathcal{H}$ satisfies

$$\nabla^{\perp}\mathcal{H} = 0,$$

where $\nabla^{\perp}$ is the normal connection. This implies in particular that the norm of $\mathcal{H}$ is constant on the surface.
For $H > 0$ we will call H-PMC surface a parallel mean curvature surface whose mean curvature vector has norm H. An H-PMC immersion is an immersion whose image is an H-PMC surface. (We will not consider the minimal case $H = 0$.)

Some classification results in 4-dimensional space forms

- In $\mathbb{M}^4(c)$, minimal surfaces with constant intrinsic curvature are either totally geodesics with $K = c$, or a part of Clifford torus with $K = 0$ in a totally geodesics $\mathbb{S}^3(c)$ of $\mathbb{S}^4(c)$, or a part of Veronese surfaces with $K = c/3$ in $\mathbb{S}^4(c)$ (Kenmotsu, 1983).
- In $\mathbb{M}^4(c)$, for H-PMC surfaces with constant intrinsic curvature, either $K = 0$ or $K = H^2 + c$. If $c \geq 0$, then the surface is either a part of product of circles with $K = 0$ or a part of 2-sphere with $K = H^2 + c$, where H denotes the norm of the mean curvature vector (Hoffman, 1973).
- When the ambient manifold is a complex 2-dimensional space form ($\mathbb{C}^2$, $\mathbb{CH}^2(c)$, $\mathbb{CP}^2(c)$), parallel mean curvature surfaces with constant intrinsic curvature are either a product of circles, or a cylinder, or a round sphere, or a slant surface or one of the Hirakawa examples (Hirakawa, 2006).

Examples (Torralbo-Urbano)

- CMC surfaces in totally geodesic hypersurfaces: since $\mathbb{M}^2(c) \times \mathbb{R}$ is isometrically immersed as a totally geodesic submanifold of $\mathbb{M}^2(c) \times \mathbb{M}^2(c)$, all H-CMC surfaces with constant intrinsic curvature yield H-PMC surfaces in $\mathbb{M}^2(c) \times \mathbb{M}^2(c)$ with constant intrinsic curvature.
- Products of constant curvature curves: if γ_1 and γ_2 are curves of constant curvatures k_1 and k_2 in $\mathbb{M}^2_c$ such that k_1 and k_2 are not both zero, then $\gamma_1 \times \gamma_2$ is an H-PMC surface with $4H^2 = k_1^2 + k_2^2$.
- Torralbo-Urbano surface in $\mathbb{H}^2 \times \mathbb{H}^2$: here we set $c = -1$. For $H \in (0, \frac{1}{2})$, Torralbo and Urbano (2012) introduced an explicit H-PMC isometric embedding of the hyperbolic plane of curvature $K = 4H^2 - 1$ that does not lie in a totally geodesic $\mathbb{H}^2 \times \mathbb{R}$.

We fix a conformal structure J on $\mathbb{M}_c^2$. Let ω be the corresponding Kähler form on $\mathbb{M}_c^2$, $\pi_j : \mathbb{M}_c^2 \times \mathbb{M}_c^2 \to \mathbb{M}_c^2$ the j-st projection ($j = 1, 2$), $\omega_1 = \pi_1^*\omega + \pi_2^*\omega$ and $\omega_2 = \pi_1^*\omega - \pi_2^*\omega$.
Let $H > 0$. Let $(\Sigma, \mathrm{d}s^2)$ be an oriented simply connected Riemannian surface. Let $\Phi : \Sigma \to \mathbb{M}_c^2 \times \mathbb{M}_c^2$ be an H-PMC isometric immersion. The manifold $\mathbb{M}_c^2 \times \mathbb{M}_c^2$ has two Kähler structures, $J_1 = (J, J)$ and $J_2 = (J, -J)$. The Kähler functions ν_j ($j = 1, 2$) of Φ for these two Kähler structures are defined by

$$\Phi^*\omega_j = \nu_j \omega_\Sigma$$

where ω_Σ is the area 2-form on Σ.

The Torralbo-Urbano correspondence

Theorem (Torralbo - Urbano, 2012)

Let $H > 0$ and Σ be a simply connected Riemannian surface. Then there exists a one to one correspondence between congruence classes of parallel mean curvature (PMC) isometric immersions $\Sigma \to \mathbb{M}^2(c) \times \mathbb{M}^2(c)$ with mean curvature vector of norm H and pairs of congruence classes of CMC H isometric immersions $\Sigma \to \mathbb{M}^2(c) \times \mathbb{R}$.

Hence, for $H \neq 0$, a pair of H-CMC isometric immersions Φ_1, Φ_2 into $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ yields an $|H|$-PMC isometric immersion Φ in $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$, and conversely, up to congruences (i.e., isometries of the ambient manifold).
Moreover these immersions are characterized as follows: $[\Phi_1] = [\Phi_2]$ if and only if Φ is a PMC immersion into a totally geodesic $\mathbb{S}^2 \times \mathbb{S}^1$ or a totally geodesic $\mathbb{H}^2 \times \mathbb{R}$. Then, in this case we have $[\Phi] = [\Phi_1]$. Here $[\,]$ denotes the congruence class of the immersion.

For given H, K, take to H-CMC isometric immersions with constant intrinsic curvature K to produce an H-PMC isometric immersion with constant intrinsic curvature K using the Torralbo-Urbano correspondence.
We have to be careful to the fact that the classification theorem in $\mathbb{M}^2(c) \times \mathbb{R}$ is about surfaces whereas the Torralbo-Urbano correspondence is about immersions.

Classification theorem in $\mathbb{S}^2 \times \mathbb{S}^2$

Theorem (D.-Domingos-Vitório)

Let $H > 0$ and $K \in \mathbb{R}$. Let Σ be an H-PMC surface with constant intrinsic curvature K in $\mathbb{S}^2 \times \mathbb{S}^2$. Then $K = 0$ and Σ is part of a product of two curves of constant curvatures.

Classification theorem in $\mathbb{H}^2 \times \mathbb{H}^2$

Theorem (D.-Domingos-Vitório)

Let $H > 0$ and $K \in \mathbb{R}$. Let Σ be an H-PMC surface with constant intrinsic curvature K in $\mathbb{H}^2 \times \mathbb{H}^2$. Then

- *either $K = 0$ and and Σ is part of a product of two curves of constant curvatures,*
- *either $K = 4H^2 - 1 < 0$ and Σ is part of*
  - *an ARL-surface in some totally geodesic $\mathbb{H}^2 \times \mathbb{R}$,*
  - *a constant intrinsic curvature helicoidal surface in some totally geodesic $\mathbb{H}^2 \times \mathbb{R}$,*
  - *a Torralbo-Urbano surface,*
  - *a surface $\mathcal{A}_{\mathrm{id}}(\Omega)$,*
  - *a surface $\mathcal{A}_{\zeta}(\Omega)$,*
  - *a surface $\mathcal{B}_{\eta}(\Omega)$,*
  - *a surface $\mathcal{B}_{m_s}(\Omega)$ for some $s \in [0, +\infty)$,*
  - *or a surface $\mathcal{B}_{\eta \circ m_s}(\Omega)$ for some $s \in [0, +\infty)$.*

Moreover, the surfaces appearing in this list are pairwise non congruent.

Sketch of the proof in $\mathbb{H}^2 \times \mathbb{H}^2$ in the case $K = 4H^2 - 1 < 0$

We set $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and we endow Ω with the metric

$$\mathrm{d}s^2 = \frac{4|\mathrm{d}z|^2}{K(z-\bar{z})^2}.$$

This is the half-plane model for the hyperbolic plane of curvature K. We assume that Ω is oriented. Its isometry group is

$$\operatorname{Isom}\Omega = \mathrm{PSL}_2(\mathbb{R}) \cup \mathrm{PSL}_2^-(\mathbb{R}),$$

where $\mathrm{PSL}_2^-(\mathbb{R}) = \{f \in \mathrm{GL}_2(\mathbb{R}) : \det f = -1\}/\{\pm \mathrm{id}\}$, with $\mathrm{PSL}_2(\mathbb{R})$ (group of orientation preserving isometries) acting on Ω by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

and $\mathrm{PSL}_2^-(\mathbb{R})$ (set of orientation reversing isometries) acting on Ω by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta}.$$

Sketch of the proof

We also set

$$D=\left\{\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}:\alpha\in\mathbb{R}^*\right\},\quad T=\left\{\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix}:\alpha\in\mathbb{R}^*,\beta\in\mathbb{R}\right\},$$

$$\xi=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \eta=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad \zeta=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}.$$

i.e.,

$$\xi(z)=-\frac{1}{z},\quad \eta(z)=-\bar{z},\quad \zeta(z)=\frac{1}{-z+1}.$$

Finally, for an immersion $\Phi:\Omega\to\mathbb{H}^2\times\mathbb{R}$ or $\Phi:\Omega\to\mathbb{H}^2\times\mathbb{H}^2$ we let

$$G_\Phi=\{f\in\operatorname{Isom}\Omega:[\Phi\circ f]=[\Phi]\}.$$

This is a subgroup of $\operatorname{Isom}\Omega$.

We let $\mathcal{X} : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ be an ARL-immersion and $\mathcal{Y} : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ an helicoidal immersion (they can be made explicit). We first determine the groups $G_{\mathcal{X}}$ and $G_{\mathcal{Y}}$:

$$\begin{aligned} G_{\mathcal{X}} &= T \cup T\eta \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{R}^*, \beta \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} -\alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{R}^*, \beta \in \mathbb{R} \end{aligned}$$

$$G_{\mathcal{Y}} = D \cup D\xi = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{R}^* \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} : \beta \in \mathbb{R}^* \right\}.$$

Sketch of the proof

To get all PMC isometric immersions, the possible choices for the pair of CMC isometric immersions are:

- $([\mathcal{X}], [\mathcal{X} \circ f])$,
- $([\mathcal{Y}], [\mathcal{X} \circ f])$,
- $([\mathcal{Y}], [\mathcal{Y} \circ f])$,

for some $f \in \operatorname{Isom} \Omega$, but we have to study which f yield the same PMC surface, i.e., the same PMC isometric immersion up to congruences and isometric reparametrisations. This gives in each case an equivalence relation on $\operatorname{Isom} \Omega$.

Sketch of the proof

For instance let us study the third case: If $f_1, f_2 \in \operatorname{Isom}\Omega$, then the corresponding PMC immersions $\mathcal{B}_{f_1}$ and $\mathcal{B}_{f_2}$ have the same image in $\mathbb{H}^2 \times \mathbb{H}^2$ up to congruences if and only if there exist $g_1, g_2 \in G_{\mathcal{Y}}$ such that

$$f_2 = g_1 \circ f_1 \circ g_2.$$

(Then we set $f_1 \approx f_2$.)
For $s \in \mathbb{R}$ we set

$$m_s = \begin{pmatrix} s & 1-s \\ -1 & 1 \end{pmatrix}.$$

Studying the equivalence classes is a linear algebra exercise.
Then each equivalence class has exactly one representative in the set

$$\{\mathrm{id}, \eta\} \cup \{m_s : s \in [0, +\infty)\} \cup \{\eta \circ m_s : s \in [0, +\infty)\}.$$

A concluding remark

The extrinsic normal curvature of an H-PMC surface in $\mathbb{H}^2 \times \mathbb{H}^2$ is given by

$$\overline{K}^{\perp} = \frac{\nu_2^2 - \nu_1^2}{2}$$

(Torralbo-Urbano).

Hence the surfaces $\mathcal{A}_{\mathrm{id}}(\Omega)$, $\mathcal{A}_{\zeta}(\Omega)$, $\mathcal{B}_{m_s}(\Omega)$ and $\mathcal{B}_{\eta\circ m_s}(\Omega)$ for $s \in [0, +\infty)$ are examples of H-PMC surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$ with non identically zero extrinsic normal curvature.

Up to our knowledge, these are the first examples of such surfaces.