Chapter 6.
Diagonalization.

1 Diagonalization Process

Most calculations are simplified if the matrices are diagonal.

1. $$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}$$
$$= \begin{pmatrix} a_1 \cdot b_1 & & & \\ & a_2 \cdot b_2 & & \\ & & \ddots & \\ & & & a_n \cdot b_n \end{pmatrix}.$$

2. $$\left| \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \right| = a_1 \cdot a_2 \cdot \cdots \cdot a_n.$$

3. $$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} a_1^{-1} & & & \\ & a_2^{-1} & & \\ & & \ddots & \\ & & & a^{-1} \end{pmatrix}.$$

4. $$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}.$$

It will therefore be interesting to have methods that allow obtaining diagonal forms for any matrix.

$$\underbrace{A}_{\text{Any matrix}} \xrightarrow[\text{diagonalization}]{} \underbrace{D}_{\text{Diagonal matrix}},$$

in such a way that we can recover for A the operations that we perform more easily on D.

Definition 1. Given $A \in \mathcal{M}_n$, we say it is a diagonalizable matrix if there exists $C \in \mathcal{M}_n$ invertible such that the matrix

$$D = C^{-1} \cdot A \cdot C$$

is a diagonal matrix. In such a case, we say that the matrix C diagonalizes the matrix A and we call it the change-of-basis matrix.

Properties 2. *Let $A \in \mathcal{M}_n$ be a diagonalizable matrix such that*

$$D = C^{-1} \cdot A \cdot C,$$

where $C, D \in \mathcal{M}_n$, with D being a diagonal matrix and C an invertible matrix. Then:

i) $|A| = |D|$.

ii) $A^n = C \cdot D^n \cdot C^{-1}$, $n \in \mathbb{N}$. *In particular, if A is invertible this property is also valid for $n \in \mathbb{Z}$, $n < 0$.*

Let $A = \begin{pmatrix} 4 & -4 & 2 \\ -3 & 5 & -2 \\ -9 & 12 & -5 \end{pmatrix}$.

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$$|A - \lambda I_3| = 2 - 5\lambda + 4\lambda^2 - \lambda^3$$

Given a matrix $A \in \mathcal{M}_{n\times n}$, to diagonalize it we must find the change-of-basis matrix C and the diagonalization D. For this, we will make the following considerations:

- We will assume that the column vectors of the change-of-basis matrix C are $v_1, \ldots, v_n \in \mathbb{R}^n$, that is,

$$C = (v_1|v_2|\ldots|v_n),$$

  and that

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

- The matrix C must have an inverse and therefore

$$\det(C) \neq 0 \Leftrightarrow \{v_1, v_2, \ldots, v_n\} \text{ are all of them independent.}$$

- It is easy to verify that

$$A \cdot C = A \cdot (v_1|v_2|\ldots|v_n) = (Av_1|Av_2|\ldots|Av_n)$$

and also that

$$C \cdot D = (\lambda_1 v_1|\lambda_2 v_2|\ldots|\lambda_n v_n).$$

- If the matrix C diagonalizes A with D being the diagonalization then, taking into account the previous point,

$$D = C^{-1} \cdot A \cdot C \Leftrightarrow A \cdot C = C \cdot D$$

$$\Leftrightarrow (Av_1|Av_2|\ldots|Av_n) = (\lambda_1 v_1|\lambda_2 v_2|\ldots|\lambda_n v_n)$$

$$\Leftrightarrow \begin{cases} Av_1 & = & \lambda_1 v_1 \\ Av_2 & = & \lambda_2 v_2 \\ & \vdots & \\ Av_n & = & \lambda_n v_n \end{cases}$$

Therefore, if we find a basis of vectors of $\mathbb{R}^n$,

$$\{v_1, v_2, \ldots, v_n\},$$

satisfying this last property then the matrix A is diagonalizable with $C = (v_1|v_2|\ldots|v_n)$ and

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Definition 3. Given $A \in \mathcal{M}_n$ we call:

- **eigenvalue of** A any real number $\lambda \in \mathbb{R}$ such that there exists some non-zero vector, $v \in \mathbb{R}^n$, such that
$$A \cdot v = \lambda v.$$
- **eigenvector of** A **associated with the eigenvalue** λ any vector $v \in \mathbb{R}^n$ such that
$$A \cdot v = \lambda v.$$
- **eigenspace of** A **associated with the eigenvalue** λ the set of all eigenvectors of A associated with the eigenvalue λ,
$$V_\lambda = \{v \in \mathbb{R}^n / A \cdot v = \lambda v\}.$$
Such set V_λ is a vector subspace of $\mathbb{R}^n$.

The concepts of eigenvector and eigenvalue have important interpretations in different iterative matrix models where they determine the stability positions of a system.

To check if λ is an eigenvalue of A we must find a non-zero vector, $v \in \mathbb{R}^n$, such that

$$A \cdot v = \lambda v.$$

Is $v = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ an eigenvector?

$$A \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

Is $\lambda = 3$ an eigenvalue?

$$|A - \lambda I_3| = 2 - 5\lambda + 4\lambda^2 - \lambda^3$$

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$$A \cdot v = \lambda v.$$

Now,

$$A \cdot v = \lambda v \Leftrightarrow A \cdot v - \lambda v = 0 \Leftrightarrow A \cdot v - \lambda I_n \cdot v = 0$$

$$\Leftrightarrow (A - \lambda I_n) \cdot v = 0,$$

$$A \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

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$$\Leftrightarrow (A - \lambda I_n) \cdot v = 0,$$

Therefore, if we call $\overline{A} = A - \lambda I_n$, what we must find is a non-zero vector $v = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that

$$\overline{A} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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$$\overline{A} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

λ is an eigenvalue of $A \Leftrightarrow \exists\, v \in \mathbb{R}^n, v \neq 0,$ such that $A \cdot v = \lambda v$

$$\Leftrightarrow \overline{A} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \text{ is indeterminate} \Leftrightarrow |\overline{A}| = 0$$

$$\Leftrightarrow |A - \lambda I_n| = 0.$$

$$|A - \lambda I_3| = 2 - 5\lambda + 4\lambda^2 - \lambda^3$$

Let V_λ be the set of all eigenvectors of A associated with the eigenvalue λ. From all the previous reasoning, it follows that

$$\begin{aligned} V_\lambda &= \{v \in \mathbb{R}^n / A \cdot v = \lambda v\} \\ &= \{v = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n / (A - \lambda I_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\} \end{aligned}$$

so V_λ is a vector subspace with implicit equations given in matrix form by

$$V_\lambda \equiv (A - \lambda I_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This is the basic technique for calculating eigenvalues and eigenvectors.

Property 4. *Given $A \in \mathcal{M}_n$:*

i) It holds that

$$\lambda \in \mathbb{R} \text{ is an eigenvalue of } A \Leftrightarrow |A - \lambda I_n| = 0$$

and, if $\lambda \in \mathbb{R}$ is an eigenvalue, then V_λ is the vector subspace of $\mathbb{R}^n$ given by

$$V_\lambda \equiv (A - \lambda I_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and therefore

$$\dim(V_\lambda) = n - \text{rango}(A - \lambda I_n).$$

ii) Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ are eigenvalues of A distinct from each other. Then, if B_1 is a basis of V_{λ_1}, B_2 is a basis of V_{λ_2},..., B_k is a basis of V_{λ_k}, we have that

$$H = B_1 \cup B_2 \cup \cdots \cup B_k$$

is an independent set.

iii) If $\lambda \in \mathbb{R}$ is an eigenvalue of the matrix A and $v \in \mathbb{R}^n$ is an eigenvector of A associated with λ then

$$A^k v = \lambda^k v.$$

Definition 5. Given $A \in \mathcal{M}_n$ we call characteristic polynomial of the matrix A the polynomial

$$p(\lambda) = |A - \lambda I_n| \in \mathbb{P}_n(\lambda)$$

and we call characteristic equation of the matrix A the equation

$$p(\lambda) = 0.$$

Remark. From all the above, it follows that:

- The eigenvalues of a matrix, $A \in \mathcal{M}_n$, are the solutions of its characteristic equation.
- A matrix can be diagonalized if we find a basis formed exclusively by eigenvectors.

We may encounter the following problems that would prevent a matrix from being diagonalizable:

1. The matrix either has no eigenvalues or has an insufficient number of them.
2. The eigenvectors of the matrix do not allow forming a basis.

Definition 6.

i) Given a polynomial, $p(\lambda) \in \mathbb{P}_n(\lambda)$, we say that $\lambda_0 \in \mathbb{R}$ is a zero of multiplicity k of $p(\lambda)$ if we can express $p(\lambda)$ in the form
$$p(\lambda) = q(\lambda) \cdot (\lambda - \lambda_0)^k,$$
where $q(\lambda) \in \mathbb{P}_{n-k}(\lambda)$ verifies that $q(\lambda_0) \neq 0$.

ii) Given $A \in \mathcal{M}_n$ and $\lambda \in \mathbb{R}$ an eigenvalue of A we say that the algebraic multiplicity of λ is k if λ is a zero of multiplicity k of the characteristic polynomial of the matrix A.

iii) Given $A \in \mathcal{M}_n$ and $\lambda \in \mathbb{R}$ an eigenvalue of A, we call geometric multiplicity of λ the dimension of the eigenspace associated with λ, V_λ, that is, $\dim(V_\lambda)$.

Property 7. *Let $A \in \mathcal{M}_n$ whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$, such that $\forall i = 1, \ldots, k$*

$$\begin{cases} n_i \text{ is the algebraic multiplicity of } \lambda_i. \\ m_i \text{ is the geometric multiplicity of } \lambda_i. \end{cases}.$$

Then it holds that

1. $n_1 + n_2 + \cdots + n_k \leq n.$
2. $1 \leq m_i \leq n_i, \ \forall i = 1, \ldots, k.$
3. *A is diagonalizable* $\Leftrightarrow \begin{cases} n_1 + n_2 + \cdots + n_k = n. \\ m_i = n_i, \ \forall i = 1, \ldots, k. \end{cases}.$

Property 8.

i) If $A \in \mathcal{M}_n$ is a diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

then A is diagonalizable, and it holds that the matrix I_n diagonalizes the matrix A, the canonical basis B_c of $\mathbb{R}^n$ is a basis of eigenvectors of A, and its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that the number of times each eigenvalue is repeated indicates its algebraic and geometric multiplicity.

ii) Every symmetric matrix is diagonalizable.

iii) If $A \in \mathcal{M}_n$ has n eigenvalues, all of them distinct, then A is diagonalizable.

iv) If the columns or the rows of $A \in \mathcal{M}_n$ all sum to the same number $r \in \mathbb{R}$, then $\lambda = r$ is an eigenvalue of A.

v) The characteristic polynomial of $A \in \mathcal{M}_2$ is

$$p(\lambda) = \lambda^2 - trace(A)\lambda + |A|.$$

vi) The characteristic polynomial of $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{M}_3$ *is*

$$\begin{aligned} p(\lambda) \; = \; & -\lambda^3 + trace(A)\lambda^2 \\ & - \left(\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| + \left| \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \right| + \left| \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \right| \right) \lambda \\ & + |A|. \end{aligned}$$

2 Study of Trends in Iterative Processes

The product and power of matrices are fundamental in formulating the most important matrix models.

Suppose we are studying a phenomenon involving several quantities $a_1, a_2, \ldots, a_k$ that vary over time. Therefore, we will have different values for them in each period, n: $a_{1,n}, a_{2,n}, \ldots, a_{k,n}$. If we arrange the value of the quantities in each period as a column vector, we have

$$P_n = \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{k,n} \end{pmatrix}.$$

Thus we have a list of k-tuples, P_0, P_1, $\ldots$, P_n, which provide the information of the phenomenon in each period.

In numerous situations we can calculate these tuples using a formula of the type

$$P_n = A^n \cdot P_0,$$

where A is a square matrix of order k called the **transition matrix**.

Consequently, the fundamental elements of these models are:

- The model will describe the situation of a certain phenomenon in successive periods. We will know the initial values which we will collect in a vector P_0 and call P_1, P_2, P_3, in general P_k, the vectors corresponding to the following periods.

- We will have a transition matrix, A, which governs the changes from one period to the next according to the matrix equations

$$P_{k+1} = AP_k \qquad \text{and} \qquad P_k = A^k P_0.$$

Studying the trend involves determining the future behavior of a model of this type, which ultimately means calculating or studying in some way the value of

$$A^k P_0$$

for large values of k

$$\lim_{k\to\infty} A^k P_0.$$

Example 9. Suppose that in a certain commercial sector three companies compete, which we will call A, B and C. From one year to the next,

	Customers of A	**Customers of B**	**Customers of C**
Switch A	80%	10%	10%
Switch B	10%	60%	20%
Switch C	10%	30%	70%

Suppose also that in the year the study began, company A had 210 customers, B had 190 and C, 320.

Assuming that year $k = 0$ is the year in which the study of the customers of the three companies began, we will call:

- $A_k =$ customers in company A after k years.
- $B_k =$ customers in company B after k years.
- $C_k =$ customers in company C after k years.

The information for each year will be grouped in a column vector :

$$P_k = \begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix}.$$

According to the problem data

$$P_0 = \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix}.$$

Applying the transition table,

- $$\underbrace{A_{k+1}}_{\text{customers in A in year } k+1} = 80\% \text{ of } A_k + 10\% \text{ of } B_k + 10\% \text{ of } C_k$$
  $$= 0.8A_k + 0.1B_k + 0.1C_k.$$

- $$\underbrace{B_{k+1}}_{\text{customers in B in year } k+1} = 10\% \text{ of } A_k + 60\% \text{ of } B_k + 20\% \text{ of } C_k$$
  $$= 0.1A_k + 0.6B_k + 0.2C_k.$$

- $$\underbrace{C_{k+1}}_{\text{customers in C in year } k+1} = 10\% \text{ of } A_k + 30\% \text{ of } B_k + 70\% \text{ of } C_k$$
  $$= 0.1A_k + 0.3B_k + 0.7C_k$$

Wusing the definition of matrix product, it is easy to realize that

$$P_{k+1} = \begin{pmatrix} 0.8A_k + 0.1B_k + 0.1C_k \\ 0.1A_k + 0.6B_k + 0.2C_k \\ 0.1A_k + 0.3B_k + 0.7C_k \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \cdot P_k.$$

Denoting $A = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix}$,

$$P_{k+1} = AP_k.$$

We therefore have,

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= AP_1 \\ P_3 &= AP_2 \\ P_4 &= AP_3 \\ &\text{etc.} \end{aligned}$$

Then, if we want to calculate P_4

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$$P_1 = AP_0$$

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$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= AP_1 \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= A(AP_0) \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 \end{aligned}$$

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$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 = A^2 P_0 \end{aligned}$$

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P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2P_0 \\
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P_4 &= AP_3
\end{aligned}
$$

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P_3 &= (AA^2)P_0 = A^3P_0 \\
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\end{aligned}$$

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Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 = A^2 P_0 \\ P_3 &= (AA^2)P_0 = A^3 P_0 \\ P_4 &= (AA^3)P_0 = A^4 P_0 \end{aligned}$$

Therefore, in general,

$$\boxed{P_k = A^k P_0}\ .$$

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Therefore, in general,

$$\boxed{P_k = A^kP_0} \ .$$

The matrix A regulates the step from one year to the next and is the **transition matrix** for this problem.

Since we know the initial distribution of customers, P_0, we can easily calculate the distribution in successive years. To do this, we compute several powers of A:

$$
\begin{aligned}
A^2 = AA &= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \\
&= \begin{pmatrix} 0.66 & 0.17 & 0.17 \\ 0.16 & 0.43 & 0.27 \\ 0.18 & 0.4 & 0.56 \end{pmatrix}. \\
A^3 = AA^2 &= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.66 & 0.17 & 0.17 \\ 0.16 & 0.43 & 0.27 \\ 0.18 & 0.4 & 0.56 \end{pmatrix} \\
&= \begin{pmatrix} 0.562 & 0.219 & 0.219 \\ 0.198 & 0.355 & 0.291 \\ 0.24 & 0.426 & 0.49 \end{pmatrix}. \\
A^4 = AA^3 &= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.562 & 0.219 & 0.219 \\ 0.198 & 0.355 & 0.291 \\ 0.24 & 0.426 & 0.49 \end{pmatrix} \\
&= \begin{pmatrix} 0.4934 & 0.2533 & 0.2533 \\ 0.223 & 0.3201 & 0.2945 \\ 0.2836 & 0.4266 & 0.4522 \end{pmatrix}.
\end{aligned}
$$

Using these calculations with equation (9) we have that

$$P_1 = AP_0 = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 219 \\ 199 \\ 302 \end{pmatrix}.$$

$$P_2 = A^2P_0 = \begin{pmatrix} 0.66 & 0.17 & 0.17 \\ 0.16 & 0.43 & 0.27 \\ 0.18 & 0.4 & 0.56 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 225.3 \\ 201.7 \\ 293 \end{pmatrix}.$$

$$P_3 = A^3P_0 = \begin{pmatrix} 0.562 & 0.219 & 0.219 \\ 0.198 & 0.355 & 0.291 \\ 0.24 & 0.426 & 0.49 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 229.71 \\ 202.15 \\ 288.14 \end{pmatrix}.$$

$$P_4 = A^4P_0 = \begin{pmatrix} 0.4934 & 0.2533 & 0.2533 \\ 0.223 & 0.3201 & 0.2945 \\ 0.2836 & 0.4266 & 0.4522 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 232.797 \\ 201.889 \\ 285.314 \end{pmatrix}.$$

On the other hand, once this model is formulated, several questions arise to be solved:

a) Is it possible to study the future trend in the distribution of customers?

b) Do equilibrium distributions exist?

In fact, the answers to the questions posed at the end of the previous example are the eigenvalues and eigenvectors of the matrix. Let's see next how to calculate them.

Example 10. Let us calculate all eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix}.$$

We start by calculating the characteristic polynomial:

$$\begin{aligned} |A - \lambda I_3| &= \\ & \left| \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 0.8-\lambda & 0.1 & 0.1 \\ 0.1 & 0.6-\lambda & 0.2 \\ 0.1 & 0.3 & 0.7-\lambda \end{pmatrix} \right| \\ = & \ (0.8-\lambda)(0.6-\lambda)(0.7-\lambda) + 0.1 \cdot 0.3 \cdot 0.1 + 0.1 \cdot 0.2 \cdot 0.1 \\ & -\big(0.1(0.6-\lambda)0.1 + 0.3 \cdot 0.2(0.8-\lambda) + 0.1 \cdot 0.1(0.7-\lambda)\big) \\ = & \ -\lambda^3 + 2.1\lambda^2 - 1.38\lambda + 0.28. \end{aligned}$$

This last expression is the characteristic polynomial of the matrix A . The characteristic equation of A is

$$-\lambda^3 + 2.1\lambda^2 - 1.38\lambda + 0.28 = 0.$$

Let us solve the characteristic equation. If we consider that we already know that $\lambda = 1$ is an eigenvalue,

$$\begin{array}{r|rrrr} & -1 & 2.1 & -1.38 & 0.28 \\ 1 & & -1 & 1.1 & -0.28 \\ \hline & -1 & 1.1 & -0.28 & \underline{|0} \end{array},$$

However, the coefficients we obtain in the last line of the previous Ruffini division (-1, 1.1 and -0.28) indicate that the equation left to solve is

$$-\lambda^2 + 1.1\lambda - 0.28 = 0$$

and this is a second-degree equation that we can solve directly by applying the corresponding formula to obtain

$$\lambda = \frac{-1.1 \pm \sqrt{1.1^2 - 4 \cdot (-1) \cdot (-0.28)}}{2 \cdot (-1)} \begin{cases} = 0.4 \\ = 0.7 \end{cases},$$

so finally, the three solutions of the characteristic equation are,

$$\begin{cases} \lambda = 1 \\ \lambda = 0.4 \\ \lambda = 0.7 \end{cases}.$$

Let us calculate the eigenspaces corresponding to each of the three eigenvalues:

- The eigenspace associated with $\lambda = 1$ is the vector subspace with implicit equations

$$V_1 \equiv (A - 1I_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_1 \equiv \begin{pmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.4 & 0.2 \\ 0.1 & 0.3 & -0.3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  Solving the system, it is easy to check that a basis for this subspace is $B_1 = \{(6, 5, 7)\}$.

- For $\lambda = 0.4$ the eigenspace is the vector subspace

$$V_{0.4} \equiv (A - 0.4I_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_{0.4} \equiv \begin{pmatrix} 0.4 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  A basis for this subspace is $B_{0.4} = \{(0, -1, 1)\}$.

- For $\lambda = 0.7$ the eigenspace is the vector subspace

$$V_{0.7} \equiv (A - 0.7I_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_{0.7} \equiv \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & -0.1 & 0.2 \\ 0.1 & 0.3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A basis for this last subspace is $B_{0.7} = \{(-3, 1, 2)\}$.

Part *ii)* of **Property 4** guarantees that by combining the elements of B_1, $B_{0.4}$ and $B_{0.7}$ we obtain a set of independent vectors

$$B = \{(6, 5, 7), (0, -1, 1), (-3, 1, 2)\}.$$

Since three independent vectors in $\mathbb{R}^3$ form a basis, B is a basis formed by eigenvectors associated, in that order, with the eigenvalues $\lambda = 1$, $\lambda = 0.4$ and $\lambda = 0.7$. Therefore, the initial matrix, A, is diagonalizable with change-of-basis matrix C and diagonalization D given by

$$C = \begin{pmatrix} 6 & 0 & -3 \\ 5 & -1 & 1 \\ 7 & 1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}.$$

2.1 The Power Method

Suppose we want to perform the calculation

$$A^k P_0$$

for some matrix $A \in \mathcal{M}_n$, the initial data n-tuple P_0 and $k \in \mathbb{N}$. Assume that the matrix A is diagonalizable. Then, we can calculate for A a basis of eigenvectors:

Eigenvector	**Associated Eigenvalue**
v_1	λ_1
v_2	λ_2
$\vdots$	$\vdots$
v_n	λ_n

Since the eigenvectors $v_1, v_2, \ldots, v_n$ form a basis of $\mathbb{R}^n$, any n-tuple can be obtained as a linear combination of them. In particular, the n-tuple P_0 can be written in the form

$$P_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for certain coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ that can be calculated by solving the corresponding system.

$$
\begin{aligned}
A^k P_0 &= A^k \left(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n\right) \\
&= A^k \alpha_1 v_1 + A^k \alpha_2 v_2 + \cdots + A^k \alpha_n v_n \\
&= \alpha_1 \underline{A^k v_1} + \alpha_2 \underline{A^k v_2} + \cdots + \alpha_n \underline{A^k v_n}.
\end{aligned}
$$

But

$$
A^k v_1 = \lambda_1^k v_1, \qquad A^k v_2 = \lambda_2^k v_2, \ldots \qquad A^k v_n = \lambda_n^k v_n
$$

so that

$$
\begin{aligned}
A^k P_0 &= \alpha_1 \underbrace{A^k v_1}_{\lambda_1^k v_1} + \alpha_2 \underbrace{A^k v_2}_{\lambda_2^k v_2} + \cdots + \alpha_n \underbrace{A^k v_n}_{\lambda_n^k v_n} \\
&= \alpha_1 \lambda_1^k v_1 + \alpha_2 \lambda_2^k v_2 + \cdots + \alpha_n \lambda_n^k v_n. \\
\Rightarrow &\boxed{A^k P_0 = \alpha_1 \lambda_1^k v_1 + \alpha_2 \lambda_2^k v_2 + \cdots + \alpha_n \lambda_n^k v_n}.
\end{aligned}
$$

As we have already commented, we see how the calculation of the matrix power A^k reduces to the simpler calculation of the numerical powers $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$.

Example 11. Suppose three investment groups, which we will call A, B and C, manage most of their capital themselves but diversify their investment by allocating a percentage to one of the other two groups. From one year to the next, they keep the investment percentages fixed according to the following table:

		invests in		
		A	**B**	**C**
Group	**A**	90%	30%	30%
	B	10%	70%	20%
	C	10%	10%	60%

Suppose that initially the capital in each group is, in millions of euros, as follows:

	Group A	**Group B**	**Group C**
Capital	17	27	21

Let us study the capital in subsequent years. To do this, we will set up a matrix model for this problem.

We will start by calling

$$P_0 = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} 17 \\ 27 \\ 21 \end{pmatrix}.$$

Then,

$$\begin{pmatrix} \mathrm{A}_{k+1} \\ \mathrm{B}_{k+1} \\ \mathrm{C}_{k+1} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.3 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{A}_k \\ \mathrm{B}_k \\ \mathrm{C}_k \end{pmatrix}$$

from which, as we have seen in previous examples, we arrive at

$$\begin{pmatrix} \mathrm{A}_k \\ \mathrm{B}_k \\ \mathrm{C}_k \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.3 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}^k \cdot \begin{pmatrix} \mathrm{A}_0 \\ \mathrm{B}_0 \\ \mathrm{C}_0 \end{pmatrix}. \tag{1}$$

If we denote

$$P_k = \begin{pmatrix} \mathrm{A}_k \\ \mathrm{B}_k \\ \mathrm{C}_k \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.3 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}$$

abbreviatedly the matrix equation (1) is written as

$$P_k = A^k P_0.$$

The characteristic polynomial of A is

$$p(\lambda) = |A - \lambda I_3| = \left| \begin{pmatrix} 0.9-\lambda & 0.1 & 0.1 \\ 0.3 & 0.7-\lambda & 0.1 \\ 0.3 & 0.2 & 0.6-\lambda \end{pmatrix} \right|$$

$$= -\lambda^3 + 2.2\lambda^2 - 1.51\lambda + 0.33.$$

The characteristic equation is:

$$\lambda^3 - 2.2\lambda^2 + 1.51\lambda - 0.33 = 0.$$

However, it is easy to check that the sum of all rows of A is equal to 1.1 so $\lambda = 1.1$ is an eigenvalue of A.

$$\begin{array}{r|rrrr} & 1 & -2.2 & 1.51 & -0.33 \\ 1.1 & & 1.1 & -1.21 & 0.33 \\ \hline & 1 & -1.1 & 0.3 & \underline{0} \end{array}$$

It remains to solve

$$1 \cdot \lambda^2 - 1.1\lambda + 0.3 = 0.$$

But this last one is a second-degree equation that can be solved directly, obtaining as a result

$$\lambda = \frac{1.1 \pm \sqrt{1.1^2 - 4 \cdot 1 \cdot 0.3}}{2 \cdot 1} \Rightarrow \lambda = 0.6 \quad \text{and} \quad \lambda = 0.5.$$

In this way we have that the matrix A has the following eigenvalues

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.6, \quad \lambda_3 = 0.5.$$

Let us next calculate the eigenvectors corresponding to the calculated eigenvalues:

- **Eigenvectors associated with $\boldsymbol{\lambda_1 = 1.1}$:** The eigenvectors associated with $\lambda_1 = 1.1$ form the eigenspace $V_{1.1}$ which has implicit equations

$$V_{1.1} \equiv (A - 1.1 I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_{1.1} \equiv \begin{pmatrix} -0.2 & 0.1 & 0.1 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.2 & -0.5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  It is easy to check that $V_{1.1} = \langle (1, 1, 1) \rangle$ and therefore $B_{1.1} = \{(1, 1, 1)\}$ is a basis for $V_{1.1}$.

Let us next calculate the eigenvectors corresponding to the calculated eigenvalues:

- **Eigenvectors associated with $\boldsymbol{\lambda_2 = 0.6}$:** The eigenvectors associated with $\lambda_2 = 0.6$ form the eigenspace $V_{0.6}$ which has implicit equations

$$V_{0.6} \equiv (A - 0.6I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_{0.6} \equiv \begin{pmatrix} 0.3 & 0.1 & 0.1 \\ 0.3 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this case $V_{0.6} = \langle(-2, 3, 3)\rangle$ and $B_{0.6} = \{(-2, 3, 3)\}$ is a basis for $V_{0.6}$.

Let us next calculate the eigenvectors corresponding to the calculated eigenvalues:

- **Eigenvectors associated with $\boldsymbol{\lambda_3 = 0.5}$:** The eigenvectors associated with $\lambda_3 = 0.5$ form the eigenspace $V_{0.5}$ which has implicit equations

$$V_{0.5} \equiv (A - 0.5I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_{0.5} \equiv \begin{pmatrix} 0.4 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now $V_{0.5} = \langle(1, 1, -5)\rangle$ and $B_{0.5} = \{(1, 1, -5)\}$ is a basis of $V_{0.5}$.

we obtain a basis of eigenvectors of A formed by the vectors

$v_1 = (1, 1, 1)$ associated with the eigenvalue $\lambda_1 = 1.1$,
$v_2 = (-2, 3, 3)$ associated with the eigenvalue $\lambda_2 = 0.6$,
$v_3 = (1, 1, -5)$ associated with the eigenvalue $\lambda_3 = 0.5$.

Let us express P_0 using this basis:

$$P_0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \Rightarrow \begin{pmatrix} 17 \\ 27 \\ 21 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 17 \\ 27 \\ 21 \end{pmatrix} = \begin{pmatrix} \alpha_1 - 2\alpha_2 + \alpha_3 \\ \alpha_1 + 3\alpha_2 + \alpha_3 \\ \alpha_1 + 3\alpha_2 - 5\alpha_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 - 2\alpha_2 + \alpha_3 = 17 \\ \alpha_1 + 3\alpha_2 + \alpha_3 = 27 \\ \alpha_1 + 3\alpha_2 - 5\alpha_3 = 21 \end{cases}$$

and solving this system we obtain $\alpha_1 = 20$, $\alpha_2 = 2$, $\alpha_3 = 1$ and therefore

$$P_0 = 20v_1 + 2v_2 + v_3 \Rightarrow \begin{pmatrix} 17 \\ 27 \\ 21 \end{pmatrix} = 20 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}.$$

Then,

$$A^k P_0 = 20A^k v_1 + 2A_k v_2 + A^k v_3 = 20 \cdot 1.1^k v_1 + 2 \cdot 0.6^k v_2 + 0.5^k v_3$$

or, equivalently,

$$\begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.3 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}^k \cdot \begin{pmatrix} 17 \\ 27 \\ 21 \end{pmatrix} = 20 \cdot 1.1^k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot 0.6^k \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} + 0.5^k \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}.$$

Now, by means of the expressions we have obtained, we can calculate the capital in each group after any number of years. For example:

- After $k = 3$ years, the capitals in each group will be determined by the tuple $P_3 = A^3 P_0$ which can be calculated via
$$P_3 = A^3 P_0 = 20 \cdot 1.1^3 v_1 + 2 \cdot 0.6^3 v_2 + 0.5^3 v_3$$
$$= 26.62 v_1 + 2 \cdot 0.432 v_2 + 0.125 v_3$$
$$= 26.62 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0.432 \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} + 0.125 \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 25.881 \\ 28.041 \\ 27.291 \end{pmatrix}.$$

Now, by means of the expressions we have obtained, we can calculate the capital in each group after any number of years. For example:

- After $k = 10$ years, the capitals in each group will be determined by the tuple $P_{10} = A^{10}P_0$ which we can calculate as:

$$\begin{aligned} P_{10} = A^{10}P_0 &= 20 \cdot 1.1^{10}v_1 + 2 \cdot 0.6^{10}v_2 + 0.5^{10}v_3 \\ &= 51.8748v_1 + 2 \cdot 0.0120932v_2 + 0.000976563v_3 \\ &= 51.8748 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0.0120932 \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} + 0.000976563 \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 51.8516 \\ 51.9121 \\ 51.9062 \end{pmatrix}. \end{aligned}$$

  Thus, after ten years, the capital in group A rises to 51.8516 million euros, in group B is 51.9121 million and in group C 51.9062 million.

If

$$P_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

then

We will now focus on the study of the trend. We assume then that we continue with a matrix model

$$P_k = A^k P_0,$$

$$A^k P_0 = \alpha_1 \lambda_1^k v_1 + \alpha_2 \lambda_2^k v_2 + \cdots + \alpha_n \lambda_n^k v_n.$$

We will now focus on the study of the trend. We assume then that we continue with a matrix model

$$P_k = A^k P_0,$$

$$A^k P_0 = \alpha_1 \underline{\lambda_1^k} v_1 + \alpha_2 \underline{\lambda_2^k} v_2 + \cdots + \alpha_n \underline{\lambda_n^k} v_n.$$

Definition 12. An eigenvalue of a matrix A is said to be the dominant eigenvalue if its absolute value is greater than that of the rest of the eigenvalues of the matrix. An eigenvector associated with the dominant eigenvalue is said to be a dominant eigenvector.

Examples 13.

1) The eigenvalues of the matrix

$$A = \begin{pmatrix} 25 & -40 & -31 \\ 2 & 1 & -2 \\ 18 & -36 & -24 \end{pmatrix}$$

are $\lambda_1 = -6$, $\lambda_2 = 5$ and $\lambda_3 = 3$. If we calculate the absolute value of these eigenvalues we have that

$$|\lambda_1| = 6, \quad |\lambda_2| = 5, \quad |\lambda_3| = 3.$$

Therefore, $\lambda_1 = -6$ is the dominant eigenvalue of the matrix A. It is possible to calculate that $V_{-6} = \langle(1, 0, 1)\rangle$. The vector $(1, 0, 1)$ is a dominant eigenvector for the matrix A.

2) The eigenvalues of the matrix

$$A = \begin{pmatrix} 25 & -38 & -31 \\ 5 & -4 & -5 \\ 14 & -28 & -20 \end{pmatrix}$$

are $\lambda_1 = -6$, $\lambda_2 = 6$ and $\lambda_3 = 1$. The corresponding absolute values are

$$|\lambda_1| = 6, \quad |\lambda_2| = 6, \quad |\lambda_3| = 1.$$

The absolute value of the first two eigenvalues coincides. In that case none of the eigenvalues has an absolute value strictly greater than that of all the others and the matrix has no dominant eigenvalue.

Suppose that in the identity (2) the eigenvalue λ_1 is the dominant eigenvalue of the matrix A and that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

Consequently, we will have that v_1 is a dominant eigenvector of A. Taking the dominant eigenvalue as a common factor on the right-hand side of (2) we have that

$$A^k P_0 = \lambda_1^k \left(\alpha_1 v_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n \right).$$

Since λ_1 is the dominant eigenvalue it is clear that

$$\left|\frac{\lambda_2}{\lambda_1}\right|, \ldots, \left|\frac{\lambda_n}{\lambda_1}\right| < 1$$

but for large values of k it is easy to check that if a number $r \in \mathbb{R}$ has an absolute value less than one ($|r| < 1$) then $r^k \approx 0$. In this way when k is large,

$$\left(\frac{\lambda_2}{\lambda_1}\right)^k \approx 0, \quad \left(\frac{\lambda_3}{\lambda_1}\right)^k \approx 0, \quad \left(\frac{\lambda_n}{\lambda_1}\right)^k \approx 0$$

and therefore when k becomes large we will have that

$$A^k P_0 = \lambda_1^k \left(\alpha_1 v_1 + \alpha_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^k}_{\approx 0} v_2 + \cdots + \alpha_n \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^k}_{\approx 0} v_n \right)$$

$$\Rightarrow \boxed{A^k P_0 \approx \lambda_1^k \alpha_1 v_1}.$$

From this we draw the following conclusions:

- For large values of k, the behavior of $A^k P_0$ depends solely on the dominant eigenvalue and the dominant eigenvector.
- Depending on the value of λ_1, the expression $\alpha_1 \lambda_1^k v_1$ will have one behavior or another. Specifically, we have:
  - If $|\lambda_1| < 1$, for large values of k we will have that $\lambda_1^k \approx 0$ and in that case
    $$\alpha_1 \lambda_1^k v_1 \approx 0.$$
    P_k in successive periods tend to vanish.
  - If $|\lambda_1| > 1$, for large values of k we will have that $\lambda^k \approx \pm\infty$ and then
    $$\alpha_1 \lambda_1^k v_1 \approx \pm\infty$$
    which means that the values in successive periods will grow or decrease without limit.
  - If $\lambda_1 = 1$, for large values of k we will have that
    $$\alpha_1 \lambda_1^k v_1 = \alpha_1 v_1$$
    and the data tuples in successive periods will tend to a constant equilibrium value given by αv_1.

- We have that for large values of k, the data in period k, P_k, can be calculated approximately by

$$P_k = A^k P_0 \approx \alpha_1 \lambda_1^k v_1.$$

In many situations it will be of interest to calculate the vector of percentages of P_k and then we will have that

$$\begin{aligned} &\text{vector of percentages of } P_k \\ &\approx \text{vector of percentages of } \alpha_1 \lambda_1^k v_1. \end{aligned}$$

But,

$$\begin{aligned} &\text{vector of percentages of } \underbrace{\alpha_1 \lambda_1^k}_{\text{number}} \underbrace{v_1}_{\text{vector}} \\ &= \text{vector of percentages of } v_1 \end{aligned}$$

so that

$$\text{percentages of } P_k \approx \text{vector of percentages of } v_1.$$

Examples 14.

1) In **Example 11** we were studying the problem of three financial groups that invest according to a certain fixed annual investment table that led to a matrix model for the calculation of the capitals of the three groups in successive periods of the form

$$P_k = \underbrace{\begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.3 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}^k}_{=A} \cdot \underbrace{\begin{pmatrix} 17 \\ 27 \\ 21 \end{pmatrix}}_{=P_0}.$$

We saw that the transition matrix A has eigenvalues

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.6, \quad \lambda_3 = 0.5$$

so the dominant eigenvalue is $\lambda_1 = 1.1$ and the corresponding dominant eigenvector is $v_1 = (1, 1, 1)$. On the other hand, the expression of the initial data tuple P_0 in the basis of eigenvectors v_1, v_2 and v_3 calculated on page 61 is

$$P_0 = \underbrace{20}_{=\alpha_1} v_1 + 2v_2 + v_3.$$

Recalling the reasoning from page 71 we have that:

- For large values of k we have that

$$P_k \approx 20 \cdot 1.1^k v_1.$$

For example:

  - After $k = 3$ years, the capital tuple, P_3, can be calculated approximately as

$$P_3 \approx 20 \cdot 1.1^3 v_1 = 26.62 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 26.62 \\ 26.62 \\ 26.62 \end{pmatrix}.$$

  - After $k = 10$ years, the capital tuple, P_{10}, can be calculated approximately as

$$P_{10} \approx 20 \cdot 1.1^{10} v_1 = 51.8748 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 51.8748 \\ 51.8748 \\ 51.8748 \end{pmatrix}.$$

It can be checked how even for not excessively high values of k the approximations provide results very similar to the exact data we obtained on page 65.

- Since the dominant eigenvalue satisfies $|\lambda_1| = |1.1| = 1.1 > 1$, we have that

$$P_k \approx \alpha_1 1.1^k v_1 = 20 \cdot 1.1^k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and the capitals of the three groups grow without limit during the course of successive years.

- The percentages represented by the capitals for year k, when k is sufficiently large, will be approximately the same as those represented by the dominant eigenvector v_1. The vector of percentages of v_1 is

$$\frac{100}{1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 33.\overline{3} \\ 33.\overline{3} \\ 33.\overline{3} \end{pmatrix}.$$

Therefore, the future trend is that:

$33.\overline{3}$ of the total capital will belong to group A.
$33.\overline{3}$ of the total capital will belong to group B.
$33.\overline{3}$ of the total capital will belong to group C.

It is observed that the trend, after a sufficiently large number of years, is that the three groups accumulate capital of the same amount.

2) If we analyze **Example 10** we have that the eigenvalues of the transition matrix are

$$\lambda_1 = 1, \quad \lambda_2 = 0.4, \quad \lambda_3 = 0.7.$$

Therefore the dominant eigenvalue is $\lambda_1 = 1$. We had also calculated the eigenvectors associated with these eigenvalues; in particular, we saw that $(6, 5, 7)$ is an eigenvector associated with the dominant eigenvalue $\lambda_1 = 1$ so $v_1 = (6, 5, 7)$ is a dominant eigenvector. If we also consider the eigenvectors associated with the other two eigenvalues we obtain the following basis of eigenvectors:

$$B = \{(6, 5, 7), (0, -1, 1), (-3, 1, 2)\}.$$

When we first formulated this example we saw that the initial data were: 120 customers in company A, 190 in B and 320 in C. This corresponded to an initial vector

$$P_0 = \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix}.$$

If we calculate the coordinates of P_0 in B we obtain the following expression:

$$P_0 = 35v_1 + 30v_2 + 15v_3.$$

Then we have that the situation after k periods is approximated by

$$A^k P_0 \approx 35 \cdot 1^k v_1 = 35v_1.$$

Since the dominant eigenvalue is equal to one, we have a stability situation in which the distribution of the companies will stabilize around the limiting value $35v_1 = 35(6, 5, 7)$. On page **??** we saw that this vector represented the percentages

$$(33.\overline{3}\%, 27.\overline{7}\%, 38.\overline{8}\%)$$

and the distribution we can expect for the future will be:

- Customers in company A = $33.\overline{3}\%$ of the total.
- Customers in company B = $27.\overline{7}\%$ of the total.
- Customers in company C = $38.\overline{8}\%$ of the total.