Capítulo 5.
Linear systems.

Topic Objectives

- Basic concepts. Expressing the solution using parameters. Linear combinations and systems.
- Gaussian elimination method.
- Cramer's rule.
- Vector subspaces. Expressing the solutions of a homogeneous linear system using linear combinations.
- Basis, dimension, and coordinates.
- Representation of lines and half-planes in $\mathbb{R}^2$. Linear programming.

1 Basic concepts

Definición 1. A linear system with m equations and n ordered variables or unknowns, $(x_1, x_2, \dots, x_n)$, is a set of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases},$$

where for each $i = 1, \dots, m$, $j = 1, \dots, n$:

- $a_{ij} \in \mathbb{R}$ is called the (i, j) coefficient of the system.
- $b_i \in \mathbb{R}$ is called the i-th constant term of the system.
- $x_1, x_2, \dots, x_n$ are called the unknowns of the system and are symbols representing an unknown value to be calculated.

We call a solution of the system any n-tuple of real numbers $(s_1, s_2, \dots, s_n) \in \mathbb{R}^n$ such that if in the system, for all $i = 1, \dots, n$, we substitute x_i by s_i, all its equations are true. Solving a system means finding the set of all its solutions.

We will say that the system is:

- **homogeneous** if $\forall i = 1, \ldots, m$, $b_i = 0$.
- **non-homogeneous** if for some $i \in \{1, \ldots, m\}$, $b_i \neq 0$.

In a system, unknown data different from the variables may appear, which we will call the parameters of the system.

Definición 2. Given the linear system

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. ,$$

we call:

- Coefficient matrix of the system to the $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m\times n}.$$

- Column of constant terms of the system to

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathcal{M}_{m\times 1}.$$

- Column of variables of the system to $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

- Matrix equation of the system or matrix form of the system to

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \Leftrightarrow A \cdot X = B.$$

- Augmented matrix of the system to the matrix

$$A^* = (A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

Ejemplos 3.

1) Consider the following system of two equations and variables (x, y, z):

$$\left\{\begin{array}{l} x + 2y + z = 1 \\ x + y - 2z = 3 \end{array}\right. .$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The matrix equation of the system is

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & -2 & 3 \end{array}\right).$$

Furthermore, it is easy to check that $(5,-2,0)$, $(10,-5,1)$, $(0,1,-1)$ are solutions of the system.

$$(5,-2,0) \text{ is a solution since taking } \begin{cases} x = 5 \\ y = -2 \\ z = 0 \end{cases}$$

$$\text{we have } \begin{cases} \begin{cases} 5 + 2(-2) + 0 = 1 \\ 5 + (-2) - 2\ 0 = 3 \end{cases} \\ \qquad\qquad \text{or} \\ \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{cases} .$$

2) The system

$$\begin{cases} x + 2y - z + 2w = 12 \\ y + z + 3w = 10 \\ x + 2y - 3z + 2w = 14 \\ 2x - 2y + 4z + 5w = 9 \end{cases}$$

has four equations and four variables.

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & -3 & 2 \\ 2 & -2 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 14 \\ 9 \end{pmatrix}.$$

It can be checked that the <u>only</u> solution is

$$(1, 2, -1, 3) \text{ or equivalently } \begin{cases} x = 1 \\ y = 2 \\ z = -1 \\ w = 3 \end{cases}.$$

3) Let us try to solve the system in the variables x and y,

$$\{x + y = 3\,.$$

If we intend to calculate x, we must know the value of y. So for example,

$$\begin{array}{ll}
\bullet \text{ if } y = 1 \text{ then} & x = 3 - y = 3 - 1 = 2, \\
& \qquad\Downarrow \\
& x = 2, y = 1 \text{ is a solution,} \\
\hline
\bullet \text{ if } y = -7 \text{ then} & x = 3 - y = 3 - (-7) = 10, \\
& \qquad\Downarrow \\
& x = 10, y = -7 \text{ is a solution,} \\
\hline
\bullet \text{ if } y = 0 \text{ then} & x = 3 - y = 3 - 0 = 3, \\
& \qquad\Downarrow \\
& x = 3, y = 0 \text{ is a solution,}
\end{array}$$

etc.

Instead of giving different values, generically we can assume that y takes a certain value α

$$\begin{cases} x + y = 3 \\ y = \alpha \end{cases}$$

whose solution is evidently

$$\begin{cases} x = 3 - \alpha \\ y = \alpha \end{cases} \quad \text{or in another way} \quad (3 - \alpha, \alpha).$$

By giving values to the parameter α we obtain all the solutions of the system. Thus we have,

$$
(3-\alpha,\alpha)\quad
\begin{array}{l}
\xrightarrow[\alpha=1]{} \quad (2,1) \text{ or in another way} \begin{cases} x=2 \\ y=1 \end{cases}, \\
\xrightarrow[\alpha=-7]{} (10,-7) \text{ or in another way} \begin{cases} x=10 \\ y=-7 \end{cases}, \\
\xrightarrow[\alpha=0]{} \quad (3,0) \text{ or in another way} \begin{cases} x=3 \\ y=0 \end{cases}, \\
\xrightarrow[\alpha=3]{} \quad (0,3) \text{ or in another way} \begin{cases} x=0 \\ y=3 \end{cases}
\end{array}
$$

and in general we will have infinitely many solutions corresponding to all other possible values of α. The set of all solutions will be

$$\{(3-\alpha,\alpha) : \alpha \in \mathbb{R}\}.$$

4)

$$\begin{cases} x+y+z+w=2 \\ x-y+z-2w=1 \end{cases},$$

$$\begin{cases} x+y+z+w=2 \\ x-y+z-2w=1 \\ z=\alpha \\ w=\beta \end{cases}.$$

Since we have assumed we know the variables z and w, we substitute them by their values and leave on the left-hand side of each equality only the variables we still do not know,

$$\begin{cases} x+y=2-\alpha-\beta \\ x-y=1-\alpha+2\beta \\ z=\alpha \\ w=\beta \end{cases}.$$

In this way, we are left to solve the system

$$\begin{cases} x+y=2-\alpha-\beta \\ x-y=1-\alpha+2\beta \end{cases}.$$

which has two variables and two equations. To do this it will suffice to add or subtract the two equations from each other as follows:

$$\begin{cases} x+y=2-\alpha-\beta \\ x-y=1-\alpha+2\beta \end{cases} \begin{array}{l} \xrightarrow[\text{adding the equations}]{} \\ \xrightarrow[\text{subtracting the equations}]{} \end{array} \begin{array}{l} 2x=3-2\alpha+\beta \\ 2y=1-3\beta \end{array}$$

and the final solution is

$$\begin{cases} x = \frac{3-2\alpha+\beta}{2} \\ y = \frac{1-3\beta}{2} \\ z = \alpha \\ w = \beta \end{cases} \quad \text{or in another form, } (\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta).$$

The system has infinitely many solutions, all of which are obtained by giving values to the parameters α and β. For example,

$$\begin{array}{l} \alpha = 1, \beta = 0 \ \rightarrow \ (\frac{1}{2}, \frac{1}{2}, 1, 0) \\ \alpha = 1, \beta = 1 \ \rightarrow \ (1, -1, 1, 1) \end{array}$$

the other solutions can be obtained in this way. Thus, the set of all solutions of the system is

$$\{(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta) \in \mathbb{R}^4 : \alpha, \beta \in \mathbb{R}\}.$$

When a system has a unique solution, as in example **4)**, it is not necessary to use any parameter to solve it. Only when a system has several solutions will we need to use parameters to express them.

Definición 4. Given the linear system with m equations and n ordered unknowns $(x_1, x_2, \ldots, x_n)$,

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dotfill \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. ,$$

the system

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dotfill \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ x_i = \alpha_i \end{array}\right. ,$$

resulting from adding to the initial system the equation $x_i = \alpha_i$, is said to have been obtained by taking the variable x_i as a parameter via the parameter α_i.

Of course, in a system it is possible to successively take different variables as parameters.

Definición 5. A linear system is said to be:

- **consistent:** If it has at least one solution.
- **inconsistent:** If it has no solution.
- **undetermined:** If it has more than one solution.
- **determined:** If it has a unique solution.
- ***d*-dimensional solution:** If in the system d variables can be selected such that the resulting system upon taking them as parameters is consistent and determined. That is, if all its solutions can be expressed by taking d variables of the system as parameters.

Schematically we have:

$$\begin{cases} \begin{array}{c}\text{Consistent}\\ \scriptstyle\text{(with solution)}\end{array} & \begin{cases} \begin{array}{c}\text{Determined}\\ \scriptstyle\text{(unique solution)}\end{array} \\ \begin{array}{c}\text{Undetermined}\\ \scriptstyle\text{(several solutions)}\end{array} & \begin{cases} \text{1-dimensional solution.} \\ \text{2-dimensional solution.} \\ \quad\vdots \\ d\text{-dimensional solution.} \end{cases} \end{cases} \\ \begin{array}{c}\text{Inconsistent}\\ \scriptstyle\text{(no solution)}\end{array} \end{cases}$$

It is evident that:

- A homogeneous system is always consistent.
- A system that needs parameters to be solved is undetermined.
- A consistent and determined system does not need any parameters (it is a 0-dimensional solution system).

Ejemplo 6. Consider the system

$$\begin{cases} x + y = 0 \\ x + y = 1 \end{cases}.$$

It is evident that it has no solution.

Teorema 7 (Rouché-Frobenius)**.** *Consider a linear system with m equations and n unknowns expressed by its matrix form:*

$$A \cdot X = B,$$

where $A \in \mathcal{M}_{m\times n}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ *and* $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$. *Then:*

i) The system is consistent $\Leftrightarrow \operatorname{rango}(A) = \operatorname{rango}(A|B)$.

ii) The system is determined $\Leftrightarrow \operatorname{rango}(A) = \operatorname{rango}(A|B) = n =$ *number of unknowns.*

iii) The system has a d-dimensional solution ($d > 0$) $\Leftrightarrow n - \operatorname{rango}(A) = d$.

Ejemplos 8.

1) The system

$$\begin{cases} 2x - y + z + w = 3 \\ y - z + w = 2 \\ 2x + z + w = -1 \\ 2x + 3y + z + 2w = 4 \end{cases}$$

has coefficient and augmented matrices equal to

$$A = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad (A|B) = \left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 2 \\ 2 & 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & 2 & 4 \end{array}\right).$$

We have

$$\text{rango}(A) = 4, \quad \text{rango}(A|B) = 4.$$

Therefore

$$\text{rango}(A) = \text{rango}(A|B) = 4 = \text{no. of variables}$$

and the system is consistent and determined.

2) The system

$$\begin{cases} 2x - y + 2z + w = 3 \\ x + 2z + w = 3 \\ y + 2z + w = 3 \\ x - 2y - 2z - w = -3 \end{cases}$$

has coefficient and augmented matrices equal to

$$A = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & -2 & -2 & -1 \end{pmatrix} \quad \text{and} \quad (A|B) = \begin{pmatrix} 2 & -1 & 2 & 1 & 3 \\ 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -2 & -2 & -1 & -3 \end{pmatrix}.$$

We have

$$\text{rango}(A) = 2, \quad \text{rango}(A|B) = 2.$$

Therefore

$$\text{rango}(A) = \text{rango}(A|B) = 2 < \text{no. of variables}$$

and the system is consistent and undetermined with a

$$n - \text{rango}(A) = 4 - 2 = 2\text{-dimensional solution.}$$

3) The system

$$\begin{cases} 2x - y + 2z + w = 3 \\ x + 2z + w = 3 \\ y + 2z + w = 3 \\ x - 2y - 2z - w = 7 \end{cases}$$

has coefficient and augmented matrices equal to

$$A = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & -2 & -2 & -1 \end{pmatrix} \quad \text{and} \quad (A|B) = \begin{pmatrix} 2 & -1 & 2 & 1 & 3 \\ 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -2 & -2 & -1 & 7 \end{pmatrix}.$$

We have

$$\operatorname{rango}(A) = 2, \quad \operatorname{rango}(A|B) = 3.$$

Therefore

$$\operatorname{rango}(A) \neq \operatorname{rango}(A|B)$$

and the system is inconsistent.

2 Gaussian elimination method

Definición 9. Given the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases},$$

we call the detailed matrix of the system the matrix

$$\begin{array}{cc} & \\ \text{numerical rows}\left\{\vphantom{\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array}}\right. & \left(\begin{array}{cccc|c} x_1 & x_2 & \dots & x_n & 0 \\ \hline a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) \\ & \underbrace{\hphantom{a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}}}_{\text{variable columns}} \end{array}.$$

Numerical rows of the detailed matrix are all its rows except the first one, and variable columns of the detailed matrix are those columns whose first element is one of the variables x_i.

Ejemplos 10.

1) $\left\{ \begin{array}{l} x_1 + 2x_2 + x_3 + 6x_4 = 1 \\ 2x_1 + 4x_2 - x_3 + 3x_4 = 1 \\ -x_1 - 2x_2 + 2x_3 - x_4 = 2 \end{array} \right.$ $\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline 1 & 2 & 1 & 6 & 1 \\ 2 & 4 & -1 & 3 & 1 \\ -1 & -2 & 2 & -1 & 2 \end{array} \right)$.

2) Corresponding to the detailed matrix $\begin{array}{c} \begin{array}{cccc} x & y & z & 0 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \end{array}$ is

$$\begin{cases} x + 2y + z = -1 \\ -x + 2y + 3z = 2 \end{cases}.$$

If we modify the detailed matrix we will also be modifying the corresponding system.

$$* \quad \begin{array}{c} \begin{array}{cccc} x & y & z & 0 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \end{array} \xrightarrow[\text{F2}\leftrightarrow\text{ F3}]{} \begin{array}{c} \begin{array}{cccc} x & y & z & 0 \end{array} \\ \left(\begin{array}{ccc|c} -1 & 2 & 3 & 2 \\ 1 & 2 & 1 & -1 \end{array}\right) \\ \Updownarrow \\ \begin{cases} -x + 2y + 3z = 2 \\ x + 2y + z = -1 \end{cases} \end{array}.$$

$$* \quad \begin{array}{c} \begin{array}{cccc} x & y & z & 0 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \end{array} \xrightarrow[\text{C1}\leftrightarrow\text{ C2}]{} \begin{array}{c} \begin{array}{cccc} y & x & z & 0 \end{array} \\ \left(\begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 2 & -1 & 3 & 2 \end{array}\right) \\ \Updownarrow \\ \begin{cases} 2y + x + z = -1 \\ 2y - x + 3z = 2 \end{cases} \end{array}.$$

$$
* \quad \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[\text{2F2}]{} \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 2 & 4 & 2 & -2 \\ -1 & 2 & 3 & 2 \end{array}\right).
$$

$$
\Updownarrow
$$

$$
\begin{cases} 2x + 4y + 2z = -2 \\ -x + 2y + 3z = 2 \end{cases}
$$

$$
* \quad \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[\text{F3=F3+F2}]{} \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ 0 & 4 & 4 & 1 \end{array}\right).
$$

$$
\Updownarrow
$$

$$
\begin{cases} x + 2y + z = -1 \\ 4y + 4z = 1 \end{cases}
$$

Definición 11. We call an elementary operation for the detailed matrix of a system any of the following actions:

1. Modify the order of the numerical rows.
2. Modify the order of the variable columns.
3. Multiply a numerical row by a non-zero number.
4. Add to a numerical row another numerical row multiplied by a number.

We must, therefore, iteratively follow these steps:

1. We select one of the variable columns.
2. In the selected column we choose a non-zero element (pivot), which must be at a height different from those selected in previous steps.
3. Using the pivot, we nullify the elements of the selected column.

Then we say that the system is in *reduced row echelon form.* A system in reduced row echelon form is solved immediately taking into account the following points:

- If after reducing the matrix through elementary operations a row appears with all its elements zero in the variable columns and the element in the constant term is non-zero, then the system will be inconsistent.

Ejemplo 12. In the following detailed matrix,

$$\left(\begin{array}{cccc|c} x & y & z & w & 0 \\ \hline 1 & 0 & -1 & 1 & -1 \\ 0 & -1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right),$$

a complete row of zeros appears accompanied in the constant terms by a non-zero element. If we write the equation corresponding to this row we would have

$$0x + 0y + 0z + 0w = 2$$

and it is evident that no valid solution exists for this equation. Therefore the system is inconsistent.

- We will solve for the variables corresponding to the columns we have reduced.
- We will take as parameters the variables corresponding to the columns that have not been reduced.

Ejemplo 13. In the following matrix,

$$\left(\begin{array}{cccc|c} \mathrm{x} & y & \mathrm{z} & w & 0 \\ \hline \overline{\underline{1}} & 1 & 0 & 1 & -1 \\ 0 & -1 & \overline{\underline{2}} & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\begin{cases} x+y+w=-1, \\ -y+2z+3w=2, \\ y=\alpha, \\ w=\beta \end{cases} \Rightarrow \begin{cases} x=-1-\alpha-\beta, \\ z=1+\frac{\alpha}{2}-\frac{3}{2}\beta, \\ y=\alpha, \\ w=\beta. \end{cases}$$

$$\Rightarrow \begin{cases} x=-1-\alpha-\beta, \\ y=\alpha, \\ z=1+\frac{\alpha}{2}-\frac{3}{2}\beta, \\ w=\beta. \end{cases}$$

Ejemplos 14.

1) Let's solve the system

$$\begin{cases} x_1 + x_2 - 2x_3 + 2x_4 + x_5 = 3 \\ 5x_1 + 3x_2 - 3x_3 + 4x_4 + 2x_5 = 4 \\ 3x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 1 \\ 2x_1 + 4x_2 - 2x_3 + x_4 = 1 \\ 3x_1 + 4x_2 - 2x_3 + x_4 = 1 \end{cases} .$$

The detailed matrix of the system is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 2 & 1 & 3 \\ 5 & 3 & -3 & 4 & 2 & 4 \\ 3 & 2 & -1 & 2 & 1 & 1 \\ 2 & 4 & -2 & 1 & 0 & 1 \\ 2 & 4 & -2 & 1 & 0 & 1 \end{array}\right) \end{array} .$$

Let's obtain the reduced row echelon matrix. To do this, we will take different pivotes, all at different heights, to nullify each column.

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 1 & -2 & 2 & 1 & 3 \\
5 & 3 & -3 & 4 & 2 & 4 \\
3 & 2 & -1 & 2 & 1 & 1 \\
2 & 4 & -2 & 1 & 0 & 1 \\
2 & 4 & -2 & 1 & 0 & 1
\end{array}\right)
\end{array}
$$

$$
\xrightarrow[\substack{\text{F3 = F3 - 5F2} \\ \text{F4 = F4 - 3F2} \\ \text{F5 = F5 - 2F2} \\ \text{F6 = F6 - 3F2}}]{\text{(pivot=}2^{\text{nd}}\text{ element of the }1^{\text{st}}\text{ column)}}
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\
\left(\begin{array}{ccccc|c}
\underline{\mathbf{1}} & 1 & -2 & 2 & 1 & 3 \\
0 & 2 & -7 & -6 & -3 & -11 \\
0 & -1 & 5 & -4 & -2 & -8 \\
0 & 2 & 2 & -3 & -2 & -5 \\
0 & 1 & 4 & -5 & -3 & -8
\end{array}\right)
\end{array}
$$

$$
\xrightarrow[\substack{\text{F2 = F2 - F6} \\ \text{F3 = F3 + 2F6} \\ \text{F4 = F4 + F6} \\ \text{F5 = F5 - 2F6}}]{\text{(pivot=}6^{\text{th}}\text{ element of the }2^{\text{nd}}\text{ column)}}
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\
\left(\begin{array}{ccccc|c}
\underline{\mathbf{1}} & 0 & -6 & 7 & 4 & 11 \\
0 & 0 & 15 & -16 & -9 & -27 \\
0 & 0 & 9 & -9 & -5 & -16 \\
0 & 0 & -6 & 7 & 4 & 11 \\
0 & \underline{\mathbf{1}} & 4 & -5 & -3 & -8
\end{array}\right)
\end{array}
$$

(pivot=4^{th} element of the 3^{rd} column)

$$\longrightarrow$$

F2 = F2 + $\frac{6}{9}$ F4

F3 = F3 - $\frac{15}{9}$ F4

F5 = F5 + $\frac{6}{9}$F4

F6 = F6 - $\frac{4}{9}$F4

$$\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\
\left(\begin{array}{ccccc|c}
\underline{\overline{\mathbf{1}}} & 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\
0 & 0 & 0 & -1 & -\frac{2}{3} & -\frac{1}{3} \\
0 & 0 & \underline{\overline{\mathbf{9}}} & -9 & -5 & -16 \\
0 & 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\
0 & \underline{\overline{\mathbf{1}}} & 0 & -1 & -\frac{7}{9} & -\frac{8}{9}
\end{array}\right)
\end{array}$$

(pivot=3^{rd} element of the 4^{th} column)

$$\longrightarrow$$

F2 = F2 + F3

F4 = F4 - 9F3

F5 = F5 + F3

F6 = F6 - F3

$$\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\
\left(\begin{array}{ccccc|c}
\underline{\overline{\mathbf{1}}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \underline{\overline{\mathbf{-1}}} & -\frac{2}{3} & -\frac{1}{3} \\
0 & 0 & \underline{\overline{\mathbf{9}}} & 0 & 1 & -13 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \underline{\overline{\mathbf{1}}} & 0 & 0 & -\frac{1}{9} & -\frac{5}{9}
\end{array}\right)
\end{array}$$

Reordering and dividing

$$\begin{cases} x_1 = 0 \\ x_2 = -\frac{5}{9} + \frac{1}{9}\alpha \\ x_3 = -\frac{13}{9} - \frac{1}{9}\alpha \\ x_4 = \frac{1}{3} - \frac{2}{3}\alpha \\ x_5 = \alpha \end{cases}$$

or in the form $\quad (0, -\dfrac{5}{9} + \dfrac{1}{9}\alpha, -\dfrac{13}{9} - \dfrac{1}{9}\alpha, \dfrac{1}{3} - \dfrac{2}{3}\alpha, \alpha)$

as a function of the parameter α.

2) Let us now study the system

$$\begin{cases} 2x_1 - x_2 + 3x_3 + 2x_4 = 1 \\ 3x_1 - 2x_2 + 4x_3 = 2 \\ x_1 + 2x_2 + x_4 = 1 \\ 3x_1 + 3x_2 + x_3 = 2 \\ 2x_1 + x_2 + x_3 - x_4 = 2 \end{cases}.$$

To do this we will try to reach its row echelon form.

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline 2 & -1 & 3 & 2 & 1 \\ 3 & -2 & 4 & 0 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 3 & 1 & 0 & 2 \\ 2 & 1 & 1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow[\substack{\text{F2 = F2 - 2F4} \\ \text{F6 = F6 + F4}}]{\text{(pivot=}4^{\text{th}}\text{ element of the }4^{\text{th}}\text{ column)}} \left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline 0 & -5 & 3 & 0 & -1 \\ 3 & -2 & 4 & 0 & 2 \\ 1 & 2 & 0 & \overline{\underline{\mathbf{1}}} & 1 \\ 3 & 3 & 1 & 0 & 2 \\ 3 & 3 & 1 & 0 & 3 \end{array}\right).$$

$$\xrightarrow[\substack{\text{F2 = F2 - 3F5} \\ \text{F3 = F3 -4F5} \\ \text{F6 = F6 -F5}}]{\text{(pivot=5}^{\text{th}}\text{ element of the 3}^{\text{rd}}\text{ column)}} \begin{array}{c} \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \end{array} \\ \left(\begin{array}{cccc|c} -9 & -14 & 0 & 0 & -7 \\ -9 & -16 & 0 & 0 & -6 \\ 1 & 2 & 0 & \underline{\overline{\mathbf{1}}} & 1 \\ 3 & 3 & \underline{\overline{\mathbf{1}}} & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}.$$

We could actually continue reducing columns but we observe that a row appears, the last one, all zeros accompanied by a non-zero constant term. Without needing to reach reduced row echelon form we deduce that the system is inconsistent.

3 Cramer's Rule (Solving systems via inverse matrix calculation)

Given any system we can always extract its matrix equation which will be of the form

$$AX = B, \qquad A \in \mathcal{M}_{m\times n},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then, it is possible to apply the rules for manipulating matrix equalities to solve the system. In fact, it would suffice to solve for X in that matrix equation to arrive at the solution in the form

$$X = A^{-1} \cdot B.$$

A system that satisfies these conditions and that can be solved by solving for the coefficient matrix is called a 'Cramer's system'.

Propiedad 15. *Consider the system with n equations and variables $(x_1, x_2, \ldots, x_n)$, given by its matrix equation*

$$A \cdot X = B,$$

where

$$A = (a_{ij})_{n\times n}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \textit{and} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then, if $|A| \neq 0$ the system is consistent and determined and its solution is

$$X = A^{-1} \cdot B = \frac{1}{|A|} \mathrm{Adj}(A)^t \cdot B. \tag{1}$$

The last expression in (1) is what is usually known as Cramer's rule. Generally, Cramer's rule is presented by developing the matrix product indicated in the property, solving for each variable individually. It then takes the following formulation:

$$x_i = \frac{\begin{vmatrix} a_{1,1} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{n,n} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}}, \qquad i = 1, \ldots, n.$$

Note that in these formulas, for the calculation of the i-th variable, the denominator contains the determinant of the coefficient matrix and the numerator contains the determinant of the matrix resulting from replacing the i-th column of the coefficient matrix with the constant terms of the system.

Ejemplo 16. Let's try to solve the system

$$\begin{cases} 3x + 2y - z = 9, \\ 2x + y + z = 2, \\ x + 2y + 2z = -1. \end{cases}$$

using the previous ideas. First we must verify that the system is indeed a Cramer system, i.e., that it has the same number of equations as variables, which is evident, and that the determinant of the coefficient matrix is different from zero. So we begin by calculating the determinant of the coefficient matrix:

$$\left|\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right| = -9 \neq 0.$$

Since the determinant is non-zero, the system is indeed a Cramer system. Now we can follow two paths to solve the system. First, we could use matrix calculation as it appears in equation (1) and, second, it would be possible to resort to the Cramer's rule equations. Let's see how we would proceed in both cases.

Method 1: Via matrix calculation. We write the system using its matrix expression,

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}.$$

Since the coefficient matrix is square and has a non-zero determinant, we can calculate its inverse and solve for it in the last equality as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}.$$

We can calculate the inverse of the coefficient matrix using any of the methods we know for this. In this case, we obtain

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}^{-1} = \frac{1}{9}\begin{pmatrix} 0 & 6 & -3 \\ 3 & -7 & 5 \\ -3 & 4 & 1 \end{pmatrix}.$$

Now we substitute the inverse by its value and perform the matrix products that appear to obtain the final result

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 0 & 6 & -3 \\ 3 & -7 & 5 \\ -3 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 15 \\ 8 \\ -20 \end{pmatrix} = \begin{pmatrix} \frac{15}{9} \\ \frac{8}{9} \\ \frac{-20}{9} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{8}{9} \\ \frac{-20}{9} \end{pmatrix} \Rightarrow \begin{cases} x = \frac{5}{3}, \\ y = \frac{8}{9}, \\ z = \frac{-20}{9}. \end{cases}$$

Method 2: Via Cramer's rule. The Cramer's formulas for solving this system would be the following:

$$x=\frac{\left|\begin{pmatrix}\mathbf{9} & 2 & -1\\ \mathbf{2} & 1 & 1\\ -\mathbf{1} & 2 & 2\end{pmatrix}\right|}{\left|\begin{pmatrix}3 & 2 & -1\\ 2 & 1 & 1\\ 1 & 2 & 2\end{pmatrix}\right|},\ y=\frac{\left|\begin{pmatrix}3 & \mathbf{9} & -1\\ 2 & \mathbf{2} & 1\\ 1 & -\mathbf{1} & 2\end{pmatrix}\right|}{\left|\begin{pmatrix}3 & 2 & -1\\ 2 & 1 & 1\\ 1 & 2 & 2\end{pmatrix}\right|},\ z=\frac{\left|\begin{pmatrix}3 & 2 & \mathbf{9}\\ 2 & 1 & \mathbf{2}\\ 1 & 2 & -\mathbf{1}\end{pmatrix}\right|}{\left|\begin{pmatrix}3 & 2 & -1\\ 2 & 1 & 1\\ 1 & 2 & 2\end{pmatrix}\right|}.$$

Note that, in the numerator, for the first variable we replace the first column of the coefficient matrix with the constant terms of the system, for the second variable we replace the second column and for the third variable the third column. Since we have already calculated the determinant of the coefficient matrix, we will only have to solve the three determinants that appear in the numerator. We have that

$$\left|\begin{pmatrix}\mathbf{9} & 2 & -1\\ \mathbf{2} & 1 & 1\\ -\mathbf{1} & 2 & 2\end{pmatrix}\right|=-15,\left|\begin{pmatrix}3 & \mathbf{9} & -1\\ 2 & \mathbf{2} & 1\\ 1 & -\mathbf{1} & 2\end{pmatrix}\right|=-8,\left|\begin{pmatrix}3 & 2 & \mathbf{9}\\ 2 & 1 & \mathbf{2}\\ 1 & 2 & -\mathbf{1}\end{pmatrix}\right|=20,$$

so finally

$$x=\frac{-15}{-9}=\frac{5}{3},\qquad y=\frac{-8}{-9}=\frac{8}{9},\qquad z=\frac{20}{-9}=-\frac{20}{9}.$$

We thus have two alternative ways to solve a Cramer system. In principle, for systems with two, three, or four equations, both methods present an equivalent difficulty and both, the one based on matrix operations and the one using Cramer's rule, can be used interchangeably.

Propiedad 17. *Consider the system with m equations and variables $(x_1, x_2, \dots, x_n)$, given by its matrix equation*

$$A \cdot X = B,$$

where

$$A = (a_{ij})_{m\times n}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Suppose the system is consistent, in which case

$$\operatorname{rango}(A \mid B) = \operatorname{rango}(A) = r.$$

Consider the minor $\tilde{A}_{r\times r}$ of A obtained as the intersection of rows $i_1, i_2, \dots, i_r$ and columns $j_1, j_2, \dots, j_r$ of A and suppose that $|\tilde{A}| \neq 0$. Then:

i) The system composed solely of the i_1-th, i_2-th, …, i_r-th equations of the initial system has the same solutions as it.

ii) The system obtained by taking as a parameter any variable that is not one of $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ is determined.

Ejemplo 18. In principle, to solve the system

$$\begin{cases} x_1 + 2x_2 - 5x_3 - x_4 = -1 \\ 2x_1 + 3x_2 - 7x_3 + x_4 = 2 \\ x_1 + x_2 - 2x_3 + 2x_4 = 3 \\ 2x_1 + x_2 - x_3 + 6x_4 = 9 \\ 3x_1 + 7x_2 - 18x_3 - 5x_4 = -6 \end{cases}$$

it would not be possible to apply Cramer's method. However we can resort to **Property 17**.

$$\text{rango}\begin{pmatrix} 1 & 2 & -5 & -1 \\ 2 & 3 & -7 & 1 \\ 1 & 1 & -2 & 2 \\ 2 & 1 & -1 & 6 \\ 3 & 7 & -18 & -5 \end{pmatrix} = \text{rango}\left(\begin{array}{cccc|c} 1 & 2 & -5 & -1 & -1 \\ 2 & 3 & -7 & 1 & 2 \\ 1 & 1 & -2 & 2 & 3 \\ 2 & 1 & -1 & 6 & 9 \\ 3 & 7 & -18 & -5 & -6 \end{array}\right) = 3$$

the system is consistent and undetermined with a 1-dimensional solution.

$$\text{rango}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix} = 3$$

and therefore said minor is regular, it has an inverse.

Let's mark in the coefficient matrix of the system the rows and columns corresponding to the selected minor,

$$
\begin{array}{r c}
 & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\
\begin{array}{r} \text{equation } 1^{\text{st}} \\ \text{equation } 2^{\text{nd}} \\ \text{equation } 3^{\text{rd}} \\ \text{equation } 4^{\text{th}} \\ \text{equation } 5^{\text{th}} \end{array} &
\begin{pmatrix}
\mathbf{1} & \mathbf{2} & -5 & -\mathbf{1} \\
\mathbf{2} & \mathbf{3} & -7 & \mathbf{1} \\
1 & 1 & -2 & 2 \\
\mathbf{2} & \mathbf{1} & -1 & \mathbf{6} \\
3 & 7 & -18 & -5
\end{pmatrix}
\end{array}.
$$

Let's eliminate the equations that do not participate in the minor and take as a parameter the variables that also do not. After this, the system becomes

$$
\begin{cases}
x_1 + 2x_2 - x_4 = 5\alpha - 1 \\
2x_1 + 3x_2 + x_4 = 7\alpha + 2 \\
2x_1 + x_2 + 6x_4 = \alpha + 9 \\
x_3 = \alpha
\end{cases}.
$$

Since, as a function of the parameter α, we already know the value of x_3, we will solve only the system

$$
\begin{cases}
x_1 + 2x_2 - x_4 = 5\alpha - 1 \\
2x_1 + 3x_2 + x_4 = 7\alpha + 2 \\
2x_1 + x_2 + 6x_4 = \alpha + 9
\end{cases}.
$$

But now this system is a Cramer system and can be solved by solving in the matrix equation which is

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix}.$$

Solving and calculating the inverse, we finally have

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & -13 & 5 \\ -10 & 8 & -3 \\ -4 & 3 & -1 \end{pmatrix} \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix} = \begin{pmatrix} 2 - \alpha \\ 3\alpha - 1 \\ 1 \end{pmatrix}.$$

The solution of the system will then be,

$$\begin{cases} x_1 = 2 - \alpha, \\ x_2 = 3\alpha - 1, \\ x_3 = \alpha, \\ x_4 = 1 \end{cases} \quad \text{or equivalently } (2 - \alpha, 3\alpha - 1, \alpha, 1).$$

4 Expressing the solution of a system using linear combinations

In this section we will see that we can also describe the solutions of a system using linear combinations.

Definición 19. Given a subset of tuples $C \subseteq \mathbb{R}^n$ and a fixed tuple $p \in \mathbb{R}^n$, the set of tuples obtained by adding the fixed tuple, p, to all tuples of C, is denoted $p + C$. That is:

$$p + C = \{p + c : c \in C\}.$$

Ejemplo 20. Consider the following set of 2-tuples

$$C = \{(1, 0), (2, 3), (-1, 4)\} \subseteq \mathbb{R}^2$$

and consider the fixed tuple $(2, -1) \in \mathbb{R}^2$.

$$\begin{aligned} p + C &= (2, -1) + \{(1, 0), (2, 3), (-1, 4)\} \\ &= \{(2, -1) + (1, 0), (2, -1) + (2, 3), (2, -1) + (-1, 4)\} \\ &= \{(3, -1), (4, 2), (1, 3)\}. \end{aligned}$$

Ejemplos 21.

1) Consider the linear system,

$$S \equiv \begin{cases} x + y + z + w = 2 \\ x - y + z - 2w = 1 \end{cases}.$$

Using the parameters α and β the solution is written in the form

$$\begin{cases} x = \frac{3-2\alpha+\beta}{2}, \\ y = \frac{1-3\beta}{2}, \\ z = \alpha, \\ w = \beta \end{cases} \quad \text{or equivalently} \quad (\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta).$$

These last two representations are what are called parametric expressions of the solution of the system.

$$(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta)$$

$$= (\underbrace{\frac{3}{2}-\alpha+\frac{1}{2}\beta}_{\substack{\text{Without parameters:} \\ \frac{3}{2} \\ \text{Part for } \alpha\text{:}-\alpha \\ \text{Part for } \beta\text{: } \frac{1}{2}\beta}} , \underbrace{\frac{1}{2}-\frac{3}{2}\beta}_{\substack{\text{Without parameters:} \\ \frac{1}{2} \\ \text{Part for } \alpha\text{:}0 \\ \text{Part for } \beta\text{: } -\frac{3}{2}\beta}} , \underbrace{\alpha}_{\substack{\text{Without parameters:} \\ 0 \\ \text{Part for } \alpha\text{:}\alpha \\ \text{Part for } \beta\text{: } 0}} , \underbrace{\beta}_{\substack{\text{Without parameters:} \\ 0 \\ \text{Part for } \alpha\text{:}0 \\ \text{Part for } \beta\text{: } \beta}})$$

$$= \underbrace{(\frac{3}{2}, \frac{1}{3}, 0, 0)}_{\text{Part without parameters}} + \underbrace{(-\alpha, 0, \alpha, 0)}_{\text{Part for } \alpha} + \underbrace{(\frac{1}{2}\beta, -\frac{3}{2}\beta, 0, 1)}_{\text{Part for } \beta}$$

$$= (\frac{3}{2}, \frac{1}{3}, 0, 0) + \alpha(-1, 0, 1, 0) + \beta(\frac{1}{2}, -\frac{3}{2}, 0, 1)$$

In summary we have that

$$(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta) =$$

$$= \underbrace{(\frac{3}{2}, \frac{1}{3}, 0, 0)}_{\text{Fixed tuple}} + \underbrace{\alpha(-1,0,1,0) + \beta(\frac{1}{2}, -\frac{3}{2}, 0, 1)}_{\substack{\text{Linear combination of} \\ (-1,0,1,0) \text{ and } (\frac{1}{2}, -\frac{3}{2}, 0, 1)}}$$

Therefore, the set of solutions of the system is:

$$\{\underbrace{(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta)}_{(\frac{3}{2},\frac{1}{3},0,0) \text{ plus an element of } \langle(-1,0,1,0),(\frac{1}{2},-\frac{3}{2},0,1)\rangle} : \alpha, \beta \in \mathbb{R}\} =$$

$$= (\frac{3}{2}, \frac{1}{3}, 0, 0) + \langle(-1,0,1,0), (\frac{1}{2}, -\frac{3}{2}, 0, 1)\rangle.$$

2) Let's calculate the solution of the following system expressing it in its parametric form and using linear combinations,

$$H \equiv \begin{cases} x_1 + x_2 - 2x_3 + 2x_4 + x_5 = 0 \\ 5x_1 + 3x_2 - 3x_3 + 4x_4 + 2x_5 = 0 \\ 3x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 0 \\ 2x_1 + 4x_2 - 2x_3 + x_4 = 0 \\ 3x_1 + 4x_2 - 2x_3 + x_4 = 0 \end{cases}.$$

Again we begin by solving the system. It is a system with a 1-dimensional solution:

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha).$$

This would be the solution of the system expressed in parametric form. Starting from it, separating the tuples corresponding to each parameter (in this case we have a single parameter α) and the terms without a parameter, we have,

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha) = (0, 0, 0, 0, 0) + \alpha(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1).$$

Therefore, the expression of the solution of the system using linear combinations is

$$(0, 0, 0, 0, 0) + \langle (0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1) \rangle$$

or, equivalently,

$$\langle (0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1) \rangle.$$

Note that:

- Homogeneous system: null fixed tuple, can be eliminated.
- Non-homogeneous system: non-null fixed tuple.

5 Vector subspaces of $\mathbb{R}^n$

5.1 Definition and properties

Definición 22. Consider the set $\mathbb{R}^n$.

We call a vector subspace of $\mathbb{R}^n$ any subset $H \subseteq \mathbb{R}^n$, formed by the solutions of a homogeneous linear system with variables $(x_1, \ldots, x_n)$,

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

We will then say that said system constitutes a set of implicit equations for the vector subspace H and we will use the notation

$$H \equiv \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

In the previous definition we can write the implicit equations using their matrix expression,

$$AX = 0,$$

where,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m\times n},\ 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m\times 1},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then in abbreviated form we have that

$$H \equiv AX = 0.$$

And then

$$H = \{p \in \mathbb{R}^n \text{ such that } Ap = 0\}.$$

A fundamental aspect in the theory of vector subspaces lies in the fact that they can always be obtained as the set of linear combinations of certain vectors.

Propiedad 23. *For every vector subspace* $H \subseteq \mathbb{R}^n$ *there exist* $v_1, v_2, \ldots, v_d \in \mathbb{R}^n$ *such that*

$$H = \langle v_1, v_2, \ldots, v_d \rangle.$$

We then say that the vector subspace H *is generated by* $v_1, v_2, \ldots, v_d$ *or also that* $v_1, v_2, \ldots, v_d$ *are a generating system of* H*.*

Ejemplos 24.

1) Consider the vector subspace of $\mathbb{R}^4$

$$H \equiv \begin{cases} x+y+z+w=0 \\ x-y+z-2w=0 \end{cases}$$

and let's try to represent it through the set of linear combinations of certain tuples. To do this, first we solve the system:

$$(\frac{-2\alpha+\beta}{2}, \frac{-3\beta}{2}, \alpha, \beta).$$

Using the properties of the sum and product of matrices,

$$\begin{gathered} (\frac{-2\alpha+\beta}{2}, \frac{-3\beta}{2}, \alpha, \beta) \\ = \alpha(-1,0,1,0) + \beta(\frac{1}{2}, -\frac{3}{2}, 0, 1) \end{gathered}$$

and therefore

$$H = \langle (-1,0,1,0), (\frac{1}{2}, -\frac{3}{2}, 0, 1) \rangle.$$

2) Let's now calculate a generating system for the vector subspace of $\mathbb{R}^5$

$$H \equiv \left\{ \begin{array}{l} x_1 + x_2 - 2x_3 + 2x_4 + x_5 = 0 \\ 5x_1 + 3x_2 - 3x_3 + 4x_4 + 2x_5 = 0 \\ 3x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 0 \\ 2x_1 + 4x_2 - 2x_3 + x_4 = 0 \\ 3x_1 + 4x_2 - 2x_3 + x_4 = 0 \end{array} \right. .$$

Again we begin by solving the system. It is a system with a 1-dimensional solution:

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha).$$

Furthermore,

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha) = \alpha(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1).$$

Therefore,

$$H = \langle (0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1) \rangle.$$

Propiedad 25. $\mathbb{R}^n$ *is a vector subspace of* $\mathbb{R}^n$*. Furthermore,*

$$\mathbb{R}^n = \langle e_1, e_2, \ldots, e_n \rangle,$$

where

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0, 0), \\ e_2 &= (0, 1, 0, \ldots, 0, 0), \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 0, 1). \end{aligned}$$

5.2 Basis and dimension

Every vector subspace of $\mathbb{R}^n$ can be written in the form

$$H = \langle v_1, v_2, \ldots, v_d \rangle.$$

Of course, it is desirable that this representation be as simple as possible and involve as few vectors as possible.

Definición 26. We call a basis of the vector subspace $H \subseteq \mathbb{R}^n$ any set of vectors $\{v_1, v_2, \ldots, v_d\} \subseteq \mathbb{R}^n$ such that:

- $H = \langle v_1, v_2, \ldots, v_d \rangle$. That is, they are a generating system of H.
- $\{v_1, v_2, \ldots, v_d\}$ is independent.

Definición 27. We call the dimension of the vector subspace $H \subseteq \mathbb{R}^n$, and denote it $\dim(H)$, the number of vectors that makes up any of its bases.

Ejemplo 28.

Consider the vector subspace of $\mathbb{R}^4$

$$H \equiv \begin{cases} x - 2z + 3w = 0 \\ y - 4z + 6w = 0 \end{cases}$$

Solving the system, it is easy to see that a generating system of H is given by

$$\{(2,4,1,0),(1,2,2,1)\}.$$

It is evident that $(2,4,1,0)$ and $(1,2,2,1)$ are independent. Consequently

$$\{(2,4,1,0),(1,2,2,1)\}$$

is a basis of H and therefore, since two vectors appear in the basis, we have that $\dim(H) = 2$.

Propiedad 29. *Consider the vector subspace $H \subseteq \mathbb{R}^n$ given by its implicit equations*

$$H \equiv \left\{ \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array} \right. . \textit{ Then:}$$

$$\dim(H) = n - \operatorname{rango} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

Ejemplo 30.

Given the vector subspace of $\mathbb{R}^4$

$$H \equiv \left\{ \begin{array}{l} x - 2z + 3w = 0 \\ y - 4z + 6w = 0 \end{array} \right. ,$$

since

$$\operatorname{rango} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 6 \end{pmatrix} = 2,$$

we have that

$$\dim(H) = 4 - 2 = 2.$$

Nota. In $\mathbb{R}^n$ we can consider, for $i = 1, \ldots, n$, the coordinate vectors (the coordinate n-tuples)

$$e_i = (0, \ldots, 0, \overset{i)}{1}, 0, \ldots, 0) \in \mathbb{R}^n.$$

It is easy to check that $B_c = \{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ since they are independent and $\mathbb{R}^n = \langle e_1, e_2, \ldots, e_n \rangle$. We will call the basis B_c the canonical basis of $\mathbb{R}^n$ and since it has n elements we conclude that

$$\dim(\mathbb{R}^n) = n.$$

Propiedad 31. *Let $H \subseteq \mathbb{R}^n$ be a vector subspace with dimension d. Then:*

1. *Any set of vectors of H with more than d elements is always dependent.*
2. *No set of vectors of H with fewer than d elements can generate all of H.*
3. *A set of vectors with d elements that is independent generates all of H (and is therefore a basis of H).*
4. *A generating system of H with d elements is independent (and therefore a basis of H).*

Propiedad 32.

1. *Any set of vectors of $\mathbb{R}^n$ with more than n elements is always dependent.*
2. *No set of vectors of $\mathbb{R}^n$ with fewer than n elements can generate all of $\mathbb{R}^n$.*
3. *A set of n vectors of $\mathbb{R}^n$ that is independent generates all of $\mathbb{R}^n$ (and is therefore a basis of $\mathbb{R}^n$).*
4. *A generating system of $\mathbb{R}^n$ with n elements is independent (and therefore a basis of $\mathbb{R}^n$).*

5.3 Coordinates with respect to a basis

Definición 33. Given the vector subspace H and the basis $B = \{v_1, v_2, \ldots, v_d\}$ of H, we call the coordinates of $v \in H$ with respect to the basis B the ordered coefficients $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d.$$

Ejemplos 34.

1) Take the canonical basis of $\mathbb{R}^n$, $B_c = \{e_1, e_2, \dots, e_n\}$. Given any vector $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ we have that

$$v = (v_1, v_2, \dots, v_n) = v_1e_1 + v_2e_2 + \cdots + v_ne_n$$

and therefore the coordinates of v with respect to B_c are $(v_1, v_2, \dots, v_n$

That is, the coordinates of any vector of $\mathbb{R}^n$ with respect to the canonical basis are itself.

2) Consider the vector subspace H that has as a basis

$$B = \{(2,0,1,0), (-1,1,1,1), (0,1,-1,1)\}.$$

Suppose we know that $(5,2,-2,2) \in H$ and that we need to calculate its coordinates with respect to B. We have that the coordinates will be the coefficients (α, β, γ) such that

$$(5,2,-2,2) = \alpha(2,0,1,0) + \beta(-1,1,1,1) + \gamma(0,1,-1,1).$$

Performing the indicated operations in this equality,

$$(5,2,-2,2) = (2\alpha - \beta, \beta + \gamma, \alpha + \beta - \gamma, \beta + \gamma)$$

$$\Leftrightarrow \begin{cases} 2\alpha - \beta = 5 \\ \beta + \gamma = 2 \\ \alpha + \beta - \gamma = -2 \\ \beta + \gamma = 2 \end{cases}$$

Solving this system we obtain that $\alpha = 2$, $\beta = -1$ and $\gamma = 3$. Therefore the coordinates of $(5,2,-2,2)$ are $(2,-1,3)$.

In fact, as we see in the previous example, the calculation of coordinates with respect to a basis always reduces to solving a linear system of equations.