Capítulo 4.
Matrices.

Topic Objectives

- Concept of matrix and data representation using matrices and tuples.
- Operations between matrices.
- Dependence and independence of tuples. Rank of a matrix.
- Calculation of the inverse matrix using elementary operations.
- Determinant of a matrix. Calculation of the inverse matrix.

1 Basic Definitions

Definition 1.

- Given $n \in \mathbb{N}$, we call an n-tuple any ordered set of n real numbers of the form
$$(a_1, a_2, \ldots, a_n).$$
  - The set of all n-tuples is denoted $\mathbb{R}^n$. Therefore,
$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in \mathbb{R}\}.$$
  - The numbers $a_1, a_2, \ldots, a_n$ are called elements of the n-tuple.
  - Two n-tuples are equal if they have the same elements in the same positions. That is,
$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \Leftrightarrow \begin{cases} a_1 = b_1, \\ a_2 = b_2, \\ \vdots, \\ a_n = b_n. \end{cases}$$
  - 2-tuples are called pairs. 3-tuples are called triples.
- We call a matrix of real numbers with m rows and n columns or of type $m \times n$ (or of order $m \times n$) a set of real numbers

ordered in the form,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $a_{ij} \in \mathbb{R}$ is the number located in row i and column j.

- The numbers that make up the matrix are called elements or coefficients of the matrix.
- The (i, j) element of the matrix is the one found in row i and column j.
- Matrices are named using capital letters (A, B, C, etc.). Given a matrix, A, its elements are generically designated by the corresponding lowercase letter and the subscripts indicating the row and column. Thus the (i, j) element of matrix A will be a_{ij}.

- The set of all matrices of type $m \times n$ is denoted as $\mathcal{M}_{m\times n}$:

$$\mathcal{M}_{m\times n} = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} / a_{ij} \in \mathbb{R}, \forall i, j \right\}.$$

  Given a matrix, A, we sometimes denote that it is of type $m \times n$ by writing $A_{m\times n}$.

- The generic matrix $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ is abbreviated by $(a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ or $(a_{ij})_{m\times n}$.

- Two matrices $A = (a_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$ and $B = (b_{ij})_{\overline{m}\times\overline{n}} \in \mathcal{M}_{\overline{m}\times\overline{n}}$ are equal if it holds that:

$$\begin{cases} m = \overline{m} \\ n = \overline{n} \\ a_{ij} = b_{ij}, \forall i = 1, \ldots, m, \forall j = 1, \ldots, n \end{cases}.$$

  That is, if they are of the same type and have the same elements located in the same place.

- The n-tuple $v = (a_1, a_2, \ldots, a_n)$ can be written in the form of a row matrix or column matrix as

$$v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{or} \qquad v = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

Examples 2.
1) On numerous occasions, a single number is sufficient to describe the state of a certain object or real-world situation. For example, if we are studying the profitability of a company in different years, in principle it will be sufficient to indicate the income it obtains at each moment:

- The first year the profits were 3 million euros.
- The second year the profits were 3.5 million euros.
- The third year the profits were 3.9 million euros.

However, a more detailed study of the company's profitability would require taking into account not only the final profits but also the volume of income and expenses:

- The first year, income 13 million, expenses 10 million.
- The second year, income 14 million, expenses 10.5 million
- The third year, income 14.3 million, expenses 10.4 million

To abbreviate, we could agree to present the data for each year in an orderly manner in the form of a row or column as follows,

$$\boxed{(\text{income,expenses})} \quad \text{or} \quad \boxed{\begin{pmatrix} \text{income} \\ \text{expenses} \end{pmatrix}}$$

In this way:

- The first year the profitability data are (13,10) or $\begin{pmatrix} 13 \\ 10 \end{pmatrix}$.

- The second year the profitability data are (14,10.5) or $\begin{pmatrix} 14 \\ 10.5 \end{pmatrix}$.

- The third year the profitability data are (14.3,10.4) or $\begin{pmatrix} 14.3 \\ 10.4 \end{pmatrix}$.

Therefore what we do is to use elements of $\mathbb{R}^2$ to represent our information.

Information of the problem		**Representation**
Two data: income and expenses at a certain moment	$\Rightarrow$	Two real numbers ordered
		$\Downarrow$
		element of $\mathbb{R}^2$ 2-tuple

The need to pose mathematical models for phenomena in which several data intervene simultaneously is what motivates the use of

- $\mathbb{R}^2$, when we have two data.
- $\mathbb{R}^3$, when we have three data.
- $\mathbb{R}^n$, when we have n data.

It is even possible that we have data that require more complex structures. For example, it could be that we had the information of three different companies for a specific year:

- Data of the first company: (13,10).
- Data of the second company: (19,15).
- Data of the third company: (17,12).

It is even possible that we have data that require more complex structures. For example, it could be that we had the information of three different companies for a specific year:

- Data of the first company: (13,10).
- Data of the second company: (19,15).
- Data of the third company: (17,12).

We can represent these data together by rows,

$$\begin{array}{rc} & \begin{array}{cc} \rotatebox{90}{income} & \rotatebox{90}{expenses} \end{array} \\ \begin{array}{r} 1^{\text{st}} \text{ company} \\ 2^{\text{nd}} \text{ company} \\ 3^{\text{rd}} \text{ company} \end{array} & \begin{pmatrix} 13 & 10 \\ 19 & 15 \\ 17 & 12 \end{pmatrix} \end{array}$$

It is even possible that we have data that require more complex structures. For example, it could be that we had the information of three different companies for a specific year:

- Data of the first company: (13,10).
- Data of the second company: (19,15).
- Data of the third company: (17,12).

or we could also have written them in columns as,

$$\begin{array}{rccc} & \text{1}^{\text{st}}\text{ company} & \text{2}^{\text{nd}}\text{ company} & \text{3}^{\text{rd}}\text{ company} \\ \begin{array}{r}\text{Income}\\ \text{Expenses}\end{array} & \multicolumn{3}{c}{\begin{pmatrix} 13 & 19 & 17 \\ 10 & 15 & 12 \end{pmatrix}} \end{array}$$

In both cases, the important thing is to fix a criterion for ordering the data. In both cases, the data ordered in the form of a table constitute what is called a matrix.

- $\begin{pmatrix} 13 & 10 \\ 19 & 15 \\ 17 & 12 \end{pmatrix}$ is a matrix with three rows and two columns,
- $\begin{pmatrix} 13 & 19 & 17 \\ 10 & 15 & 12 \end{pmatrix}$ is a matrix with two rows and three columns.

2) Take the matrix

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & -12 & 4 \end{pmatrix}.$$

It is of order 2×3 ,

- the $(2,1)$ element of A is 0,
- the $(2,3)$ element of A is 4,
- the $(3,1)$ element of A does not exist,
- etc.

Likewise, we have that A is an element of the set of all matrices of type 2×3, that is, $A \in \mathcal{M}_{2\times 3}$.

Definition 3 (Basic concepts about matrices)**.**

- Given $A = (a_{ij})_{m\times n}$, we call any matrix obtained by deleting rows and/or columns from A a submatrix of A.
- A matrix of the form $\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}_{1\times n}$ of type $1 \times n$ that has only one row is called a row matrix.

Example 4. The matrices $\begin{pmatrix} 2 & -1 & 0 & 1 \end{pmatrix}_{1\times 4}$, $\begin{pmatrix} -1 & 4 & 12 \end{pmatrix}_{1\times 3}$ or $\begin{pmatrix} 2 & 4 \end{pmatrix}_{1\times 2}$ are row matrices.

- A matrix of the form $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_{n\times 1}$ of type $n \times 1$ that has only one column is called a column matrix.

Example 5. The matrices $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 4 \end{pmatrix}_{4\times 1}$ and $\begin{pmatrix} 2 \\ 6 \end{pmatrix}_{2\times 1}$ are column matrices.

- Given the n-tuples $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$:
  - The matrix obtained by grouping v_1, v_2, ..., v_n by columns is denoted
  $$\left(v_1 \mid v_2 \mid \cdots \mid v_n\right)$$
  and will have m rows and n columns.
  - The matrix obtained by grouping v_1, v_2, ..., v_n by rows is denoted
  $$\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_n \end{pmatrix}$$
  which will have n rows and m columns. That is, it is of type $n \times m$.

Example 6. Let $v_1 = (2, 3, -1, 0)$, $v_2 = (6, 2, 3, 3)$, $v_3 = (6, 4, -9, -1)$. The block matrix obtained by grouping v_1, v_2 and v_3 by columns is

$$\begin{pmatrix} v_1 \mid v_2 \mid v_3 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 3 & 2 & 4 \\ -1 & 3 & -9 \\ 0 & 3 & -1 \end{pmatrix} \in \mathcal{M}_{4\times 3}.$$

The block matrix we obtain by grouping v_1, v_2 and v_3 by rows is

$$\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline v_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 & 0 \\ 6 & 2 & 3 & 3 \\ 6 & 4 & -9 & -1 \end{pmatrix} \in \mathcal{M}_{3\times 4}.$$

- A matrix with n rows and n columns (of type $n \times n$) is said to be a square matrix of order n. The set of all square matrices of order n is denoted by $\mathcal{M}_n$.
- Given $A \in \mathcal{M}_{m\times n}$ we call the transpose of A and denote it by A^t, the matrix whose first row is the first column of A, whose second row is the second column of A, $\ldots$, whose n-th row is the n-th column of A. The following facts are evident:
  - $A \in \mathcal{M}_{m\times n} \Rightarrow A^t \in \mathcal{M}_{n\times m}$.
  - $(A^t)^t = A$.
  - $A = (a_{ij})_{m\times n} \Rightarrow A^t = (a_{ji})_{n\times m}$. That is, the element located in A at position (i, j) when transposed moves to position (j, i).

Examples 7.

1)

$$\begin{array}{ccc} \text{column 1} & \text{column 2} & \text{column 3} \end{array}$$

$$\begin{pmatrix} 2 & 3 & 7 \\ 1 & 6 & 4 \end{pmatrix}^t = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 7 & 4 \end{pmatrix} \begin{array}{l} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array}.$$

See that the initial matrix is of type 2×3 and when transposing we obtain one of type 3×2. It is also evident that if we transpose the last matrix again we will get the first one back:

$$\begin{pmatrix} 2 & 3 & 7 \\ 1 & 6 & 4 \end{pmatrix}^{tt} = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 7 & 4 \end{pmatrix}^t = \begin{pmatrix} 2 & 3 & 7 \\ 1 & 6 & 4 \end{pmatrix}.$$

- We call the zero matrix of type $m \times n$ and denote it by $0_{m\times n}$ or simply 0, the matrix:
$$0_{m\times n} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{m\times n} \in \mathcal{M}_{m\times n}.$$
That is, the matrix $0_{m\times n}$ is the matrix of type $m \times n$ that has all its elements equal to zero.

Example 8. The zero matrix of type 2×4 is
$$0_{2\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The zero matrix of type 3×3 is
$$0_{3\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In general, we can also consider
$$0_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0_{4\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Definition 9 (Basic concepts for square matrices)**.**

- We call the main diagonal of the matrix $A_{n\times n} = (a_{ij})_{n\times n} \in \mathcal{M}_n$ the row matrix $\begin{pmatrix} a_{11} & a_{22} & \dots & a_{nn}\end{pmatrix}$. The main diagonal is therefore the row matrix formed by the elements of A that are boxed in the following representation:

$$A = \begin{pmatrix} \boxed{a_{11}} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & \boxed{a_{22}} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & \boxed{a_{33}} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \boxed{a_{nn}} \end{pmatrix}.$$

- We call the trace of $A = (a_{ij})_{n\times n} \in \mathcal{M}_n$ and denote it by $tr(A)$ the sum of the elements of the main diagonal:

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

Example 10. Let $A = \begin{pmatrix} 1 & 2 & -6 \\ -5 & 2 & 4 \\ 3 & 7 & 9 \end{pmatrix}$ then we have that

- the main diagonal of A is $\begin{pmatrix} 1 & 2 & 9 \end{pmatrix}$.
- the trace of A is $\mathrm{tr}(A) = 1 + 2 + 9 = 12$.

- We say that $(a_{ij})_{n\times n} \in \mathcal{M}_n$ is:
  - **upper triangular** if all the elements below the main diagonal are zero,
  - **lower triangular** if all the elements above the main diagonal are zero,
  - **diagonal** if all the elements outside the main diagonal are zero.

Examples 11.

1) $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are upper triangular matrices.

2) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 2 & -1 & 0 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are lower triangular matrices.

3) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are diagonal matrices.

- We call the identity matrix of order n and denote it by I_n the square matrix of order n that is diagonal and such that all the elements of its main diagonal are equal to 1. That is,

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in \mathcal{M}_n.$$

Example 12.

- The identity matrix of order 1 is $I_1 = (1)$,

- the identity matrix of order 2 is $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

- the identity matrix of order 3 is $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- We say that $(a_{ij})_{n\times n}$ is symmetric if $A = A^t$ or, equivalently:

$$a_{ij} = a_{ji}, \qquad \forall i, j = 1, \ldots, n.$$

- We say that $(a_{ij})_{n\times n}$ is antisymmetric if it satisfies that:

$$a_{ij} = -a_{ji}, \qquad \forall i, j = 1, \ldots, n.$$

Note that if the above holds, taking $i = j$ we obtain that

$$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0, \quad \forall i = 1, \ldots, n$$

Examples 13.

1) Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ we have that

$$A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix} = A$$

and therefore A is a symmetric matrix. See that the elements (i, j) and (j, i) of the matrix coincide:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

2) The matrix $B = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}$ is antisymmetric since, if we denote $B = (b_{ij})_{3\times 3}$, we have that

* $b_{12} = -2 = -b_{21}$.
* $b_{13} = 3 = -b_{31}$.
* $b_{23} = -4 = -b_{32}$.
* $b_{11} = 0, b_{22} = 0, b_{33} = 0$.

$$\begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}.$$

The elements located at the ends of the same arrow must be opposites and those on the diagonal (enclosed in a circle) must be zero.

Remark. Diagonal matrices are usually denoted, in order to abbreviate writing, by indicating only the elements of their main diagonal. Thus for example:

$$\text{The matrix } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ can be written as } \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}.$$

In generic form, the diagonal matrix $A \in \mathcal{M}_n$ whose main diagonal is $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ will be denoted by

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}_{n \times n}.$$

2 Operations with matrices

2.1 Matrix addition

Definition 14. Given two matrices of the same type $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$ we define the sum of A and B as the matrix $A + B \in \mathcal{M}_{m\times n}$ determined by:

$$A + B = (a_{ij} + b_{ij})_{m\times n}.$$

That is:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}.$$

Examples 15.

1) $\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix} + \begin{pmatrix}2 & 1\\ -1 & 4\end{pmatrix} = \begin{pmatrix}3 & 3\\ 2 & 5\end{pmatrix}$.

2) $\begin{pmatrix}1 & 2 & 0 & 9\\ -1 & 4 & 6 & 6\\ 2 & 1 & 2 & 4\end{pmatrix} + \underbrace{\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}}_{=0_{3\times 4}} = \begin{pmatrix}1 & 2 & 0 & 9\\ -1 & 4 & 6 & 6\\ 2 & 1 & 2 & 4\end{pmatrix}$.

3) $\begin{pmatrix}1 & 2 & 0\\ -1 & 4 & 6\\ 2 & 1 & 2\end{pmatrix} + \begin{pmatrix}-1 & -2 & 0\\ 1 & -4 & -6\\ -2 & -1 & -2\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = 0_{3\times 3}$.

4) $\begin{pmatrix}1 & 2 & 1\\ 2 & 0 & -1\end{pmatrix} + \begin{pmatrix}0 & -1\\ 4 & 6\end{pmatrix}$ is an operation that cannot be performed because the two matrices are not of the same type.

5)

- $(2, 3, -1) + (6, 0, 2) = (8, 3, 1)$.
- $(3, -2) - (-3, 2) = (0, 0)$.
- $(4, 3, 2, 1, 1) + (-2, 4, 3, 0, -1) = (2, 7, 5, 1, 0)$.
- $(3, 2, 1) + (2, 4, 6, 2)$ is an operation that cannot be performed since we have different types.

Properties 16. $\forall A, B, C \in \mathcal{M}_{m\times n}$:

1. *Commutative property:* $A + B = B + A$.
2. *Associative property:* $A + (B + C) = (A + B) + C$.
3. $A + 0 = A$ *(where* $0 = 0_{m\times n}$*).*
4. *Given* $A = (a_{ij})_{m\times n}$ *we define the opposite matrix of* A *as*
$$-A = (-a_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$$
and then it is verified that:
$$A + (-A) = 0.$$
5. $(A + B)^t = A^t + B^t$.
6. A *is antisymmetric* $\Leftrightarrow A^t = -A$

Remark (Matrix subtraction)**.** In the same way that the sum is defined, it is equally easy to introduce matrix subtraction. In fact, as we see below, we can define subtraction from the sum.

Given two matrices $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$ we will define the subtraction or difference between A and B as the matrix
$$A - B = A + (-B) = (a_{ij} - b_{ij})_{m\times n} \in \mathcal{M}_{m\times n}.$$

Examples 17.

1) $-\begin{pmatrix}1 & 2 & 1\\ 3 & 1 & 3\end{pmatrix} = \begin{pmatrix}-1 & -2 & -1\\ -3 & -1 & -3\end{pmatrix}.$

2) $-0_{2\times 2} = -\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}-0 & -0\\ -0 & -0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 0_{2\times 2}.$

3) $\begin{pmatrix}1 & 2 & 4\\ 2 & 0 & 6\end{pmatrix} - \begin{pmatrix}2 & -1 & 0\\ 3 & 2 & 3\end{pmatrix} = \begin{pmatrix}-1 & 3 & 4\\ -1 & -2 & 3\end{pmatrix}.$

4) $-(3, 2, -1) = (-3, -2, 1).$

5) If $A = \begin{pmatrix}0 & 7\\ -7 & 0\end{pmatrix}$ then, $A^t = \begin{pmatrix}0 & -7\\ 7 & 0\end{pmatrix} = -\begin{pmatrix}0 & 7\\ -7 & 0\end{pmatrix} = -A$, therefore A is an antisymmetric matrix.

2.2 Product of matrices by a real number

Definition 18. Given a matrix $A = (a_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$ and a real number $r \in \mathbb{R}$, we define the product of r by A and denote it as $r \cdot A$ or $A \cdot r$ as:

$$r \cdot A = A \cdot r = (r \cdot a_{ij})_{m\times n} \in \mathcal{M}_{m\times n}.$$

That is:

$$r \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} r \cdot a_{11} & r \cdot a_{12} & \dots & r \cdot a_{1n} \\ r \cdot a_{21} & r \cdot a_{22} & \dots & r \cdot a_{2n} \\ \vdots & \vdots & & \vdots \\ r \cdot a_{m1} & r \cdot a_{m2} & \dots & r \cdot a_{mn} \end{pmatrix}.$$

Note that if we multiply a matrix of type $m \times n$ by a real number we obtain as a result a matrix of type $m \times n$.

Examples 19.

1) $2 \cdot \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 2 & 2 \\ 4 & 12 & 0 & -2 \end{pmatrix}$.

2) $8 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{3\times 2}$.

3) $-3 \cdot \begin{pmatrix} 1 & 2 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ 3 & -18 \end{pmatrix}$.

Properties 20. $\forall r, s \in \mathbb{R},\ \forall A, B \in \mathcal{M}_{m\times n}$:

1. Distributive property: $r \cdot (A + B) = r \cdot A + r \cdot B$.

2. Distributive property: $(r + s) \cdot A = r \cdot A + s \cdot A$.

3. $1 \cdot A = A$.

4. $(-1) \cdot A = -A$.

5. $(r \cdot s) \cdot A = r \cdot (s \cdot A)$.

6. $r \cdot 0 = 0$, $0 \cdot A = 0$.

7. $(r \cdot A)^t = r \cdot A^t$.

2.3 Product of two matrices

Definition 21. Given $A = (a_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$ and $B = (b_{ij})_{n\times p} \in \mathcal{M}_{n\times p}$ we define the product of A and B and denote it $A \cdot B$, as the matrix

$$A\cdot B = (a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj})_{\substack{i=1,\ldots,m \\ j=1,\ldots,p}} \in \mathcal{M}_{m\times p}.$$

That is, in the position (i, j) of the matrix $A \cdot B$ is the element

row i of A

$$a_{i1}\; b_{1j} \;+\; a_{i2}\; b_{2j} \;+\; a_{i3}\; b_{3j} \;+\; \ldots \;+\; a_{in}\; b_{nj} \quad ,$$

column j of B

which is obtained as the product of row i of A by column j of B.

Examples 22.

1) $\begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix}_{2\times 3} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 1 \end{pmatrix}_{3\times 2} =$

$$= \begin{pmatrix} \underbrace{\underbrace{\begin{pmatrix} 2 & 1 & 2 \end{pmatrix}}_{\text{row } 1} \cdot \underbrace{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}_{\text{column } 1}}_{=2\cdot 2+1\cdot(-1)+2\cdot 3=9} & \underbrace{\underbrace{\begin{pmatrix} 2 & 1 & 2 \end{pmatrix}}_{\text{row } 1} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{column } 2}}_{=2\cdot 1+1\cdot 1+2\cdot 1=5} \\ \underbrace{\underbrace{\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}}_{\text{row } 2} \cdot \underbrace{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}_{\text{column } 1}}_{=0\cdot 2+(-1)\cdot(-1)+0\cdot 3=1} & \underbrace{\underbrace{\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}}_{\text{row } 2} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{column } 2}}_{=0\cdot 1+(-1)\cdot 1+0\cdot 1=-1} \end{pmatrix}_{2\times 2} = \begin{pmatrix} 9 & 5 \\ 1 & -1 \end{pmatrix}_{2\times 2}.$$

2) $\begin{pmatrix} 1 & 3 & 6 \\ 3 & 9 & -1 \end{pmatrix}_{2\times 3} \cdot I_3 = \begin{pmatrix} 1 & 3 & 6 \\ 3 & 9 & -1 \end{pmatrix}_{2\times 3} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} =$

$$= \begin{pmatrix} \underbrace{\underbrace{\begin{pmatrix} 1 & 3 & 6 \end{pmatrix}}_{\text{row } 1} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{column } 1}}_{=1\cdot1+3\cdot0+6\cdot0=1} & \underbrace{\underbrace{\begin{pmatrix} 1 & 3 & 6 \end{pmatrix}}_{\text{row } 1} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{column } 2}}_{=1\cdot0+3\cdot1+6\cdot0=3} & \underbrace{\underbrace{\begin{pmatrix} 1 & 3 & 6 \end{pmatrix}}_{\text{row } 1} \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{column } 3}}_{=1\cdot0+3\cdot0+6\cdot1=6} \\ \underbrace{\underbrace{\begin{pmatrix} 3 & 9 & -1 \end{pmatrix}}_{\text{row } 2} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{column } 1}}_{=3\cdot1+9\cdot0+(-1)\cdot0=3} & \underbrace{\underbrace{\begin{pmatrix} 3 & 9 & -1 \end{pmatrix}}_{\text{row } 2} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{column } 2}}_{=3\cdot0+9\cdot1+(-1)\cdot0=9} & \underbrace{\underbrace{\begin{pmatrix} 3 & 9 & -1 \end{pmatrix}}_{\text{row } 2} \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{column } 3}}_{=3\cdot0+9\cdot0+(-1)\cdot1=-1} \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} 1 & 3 & 6 \\ 3 & 9 & -1 \end{pmatrix}_{2\times 3}.$$

3) $\underbrace{\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}}_{0_{1\times 3}} \cdot \begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 3 & 2 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 0 & 0 \end{pmatrix}_{1\times 2}.$

4) $\begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix}_{3\times 1} \cdot \begin{pmatrix} 1 & 3 & 3 \end{pmatrix}_{1\times 3} = \begin{pmatrix} 1 & 3 & 3 \\ 9 & 27 & 27 \\ 1 & 3 & 3 \end{pmatrix}_{3\times 3}.$

5) $\begin{pmatrix}1 & 3 & 3\end{pmatrix}_{1\times 3} \cdot \begin{pmatrix}1\\9\\1\end{pmatrix}_{3\times 1} = \begin{pmatrix}1\cdot 1+3\cdot 9+3\cdot 1\end{pmatrix}_{1\times 1} = \begin{pmatrix}31\end{pmatrix}_{1\times 1} \equiv$ 31.

6) $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}_{2\times 2} \cdot \begin{pmatrix}1 & 3\\2 & 1\\3 & -1\end{pmatrix}_{3\times 2}$ is an operation that cannot be performed because the types of matrices do not match.

Remark. Note that the product of a matrix of type $m \times n$ by a matrix $n \times p$ provides a matrix $m \times p$. Schematically we have:

$$m \quad \times \quad n \quad \cdot \quad n \quad \times \quad p$$

must be equal

result type $m \times p$

Properties 23.

1. $\forall A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{n\times p}, C \in \mathcal{M}_{p\times r}$
$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

2. $\forall A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{n\times p}, r \in \mathbb{R}$
$$A \cdot (r \cdot B) = (r \cdot A) \cdot B = r \cdot (A \cdot B).$$

3. $\forall A, B \in \mathcal{M}_{m\times n}, C \in \mathcal{M}_{n\times p}$
$$(A + B) \cdot C = A \cdot C + B \cdot C.$$

4. $\forall A \in \mathcal{M}_{m\times n}, B, C \in \mathcal{M}_{n\times p}$
$$A \cdot (B + C) = A \cdot B + A \cdot C.$$

5. $\forall A \in \mathcal{M}_{m\times n}$
$$\begin{array}{ll} I_m \cdot A = A & , \; 0_{p\times m} \cdot A_{m\times n} = 0_{p\times n} \\ A \cdot I_n = A & , \; A_{m\times n} \cdot 0_{n\times p} = 0_{m\times p} \end{array}.$$

6. $\forall A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{n\times p}$
$$(A \cdot B)^t = B^t \cdot A^t.$$

7. $\forall A, B \in \mathcal{M}_n, A \cdot B \in \mathcal{M}_n.$

8. Given $A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}_{n\times n}$, $B = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}_{n\times n}$, *diagonal matrices of* $\mathcal{M}_n$*, we have that:*

$$A \cdot B = \begin{pmatrix} a_1 \cdot b_1 & & & \\ & a_2 \cdot b_2 & & \\ & & \ddots & \\ & & & a_n \cdot b_n \end{pmatrix}_{n\times n}.$$

Example 24. If we consider the matrices

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix},$$

we can perform their product in the usual way. However, since both are diagonal, it is sufficient to multiply the elements of the main diagonal so that

$$A \cdot B = \begin{pmatrix} 3 \cdot 2 & 0 & 0 & 0 \\ 0 & (-1)\cdot(-3) & 0 & 0 \\ 0 & 0 & 4 \cdot 1 & 0 \\ 0 & 0 & 0 & 2 \cdot 5 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

Remark. Given two matrices $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{p\times r}$, we may ask when it will be possible to perform both the operation $A \cdot B$ and the operation $B \cdot A$. We have that:

$$\left.\begin{array}{l}\text{If it is possible to perform } A_{m\times n} \cdot B_{p\times r} \Rightarrow n = p \\ \text{If it is possible to perform } B_{p\times r} \cdot A_{m\times n} \Rightarrow r = m\end{array}\right\} \Rightarrow \left\{\begin{array}{l} A \in \mathcal{M}_{m\times n} \\ B \in \mathcal{M}_{n\times m}\end{array}\right.$$

$$\text{in which case } \left\{\begin{array}{c} A \cdot B \in \mathcal{M}_{m\times m} \\ B \cdot A \in \mathcal{M}_{n\times n}\end{array}\right. .$$

In particular, if $A, B \in \mathcal{M}_n$ we obtain $A \cdot B, B \cdot A \in \mathcal{M}_n$.

Examples 25.

1) $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}_{2\times 3} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_{3\times 1} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_{3\times 1} \cdot \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}_{2\times 3}$ cannot be calculated.

2) If $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$ since they are of type 2×3 and 3×2, we can calculate both $A \cdot B$ and $B \cdot A$. However we have that:

$\star$ $A \cdot B = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 4 & -1 \end{pmatrix}$.

$\star$ $B \cdot A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 4 \\ 7 & 3 & 2 \end{pmatrix}$.

We thus obtain that $A \cdot B \neq B \cdot A$ and furthermore they are not even matrices of the same type.

3) If $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ then we can calculate both $A \cdot B$ and $B \cdot A$ and furthermore $A \cdot B, B \cdot A \in \mathcal{M}_2$, however:

$\star$ $A \cdot B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ -1 & -1 \end{pmatrix}$.

$\star$ $B \cdot A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

So in this case $A \cdot B$ and $B \cdot A$ are of the same type but are different matrices.

4) Given any matrix $A \in \mathcal{M}_n$ it will be verified that

$$A \cdot I_n = I_n \cdot A = A$$

so in this situation the products $A \cdot I_n$ and $I_n \cdot A$ do coincide.

2.3.1 Power of matrices

Given a matrix $A \in \mathcal{M}_{m\times n}$ it will be possible to perform the product

$$A_{m\times n} \cdot A_{m\times n}$$

only when $m = n$ in which case A would be a square matrix of order n. In those cases where A is not a square matrix it will never be possible to perform the operation $A \cdot A$.

Definition 26. Given $A \in \mathcal{M}_n$ we define, for $k \in \mathbb{N}$,

$$A^k = A \cdot A \cdot A \cdot \overset{k)}{\cdots} \cdot A \in \mathcal{M}_n.$$

Examples 27.
1) Given

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

Being square we can calculate its powers.

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 \\ 5 & 0 & 0 \\ 5 & 2 & -1 \end{pmatrix}.$$

We can also calculate A^3.

$$A^3 = A \cdot A \cdot A = A \cdot A^2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 1 & 2 \\ 5 & 0 & 0 \\ 5 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 15 & 2 & 4 \\ 10 & 5 & 0 \\ 10 & 4 & 3 \end{pmatrix}.$$

In reality, to calculate the successive powers of the matrix A (A^4, A^5, etc.), we can repeat this process by multiplying the last power obtained again by the matrix A to thus obtain the next power.

For example, if we repeat the process once more we obtain A^4:

$$A^4 = A \cdot A^3 =$$

$$= \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{pmatrix}}_{\text{The matrix } A,} \cdot \underbrace{\begin{pmatrix} 15 & 2 & 4 \\ 10 & 5 & 0 \\ 10 & 4 & 3 \end{pmatrix}}_{\substack{\text{multiplied by the} \\ \text{previous power, } A^3,}} = \underbrace{\begin{pmatrix} 40 & 9 & 8 \\ 25 & 5 & 10 \\ 30 & 3 & 11 \end{pmatrix}}_{\substack{\text{provides the next} \\ \text{power, } A^4.}} .$$

When the matrix for which we want to calculate the power is diagonal, then the power calculation is simplified notably.

Properties 28.

1. *Given* $A \in \mathcal{M}_n$ *and* $k, p \in \mathbb{N}$
$$A^k \cdot A^p = A^p \cdot A^k = A^{k+p}.$$

2. *Given the diagonal matrix* $A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}_{n\times n} \in$ $\mathcal{M}_n$ *and* $k \in \mathbb{N}$ *it is verified that:*
$$A^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}_{n\times n}.$$

Examples 29.

1) $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^4 = \begin{pmatrix} 2^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & (-1)^4 \end{pmatrix} = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2) Given $A \in \mathcal{M}_n$ it is verified that

$$\begin{aligned} A^3 \cdot A^2 &= (A \cdot A \cdot A) \cdot (A \cdot A) = A \cdot A \cdot A \cdot A \cdot A = A^5, \\ A^2 \cdot A^3 &= (A \cdot A) \cdot (A \cdot A \cdot A) = A \cdot A \cdot A \cdot A \cdot A = A^5. \end{aligned}$$

3)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}^3 = \begin{pmatrix} 3 & 6 & 2 \\ 1 & -1 & 3 \\ 5 & 8 & 5 \end{pmatrix} \neq \begin{pmatrix} 1^3 & 2^3 & 0^3 \\ 0^3 & (-1)^3 & 1^3 \\ 1^3 & 2^3 & 1^3 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 0 \\ 0 & -1 & 1 \\ 1 & 8 & 1 \end{pmatrix}.$$

2.4 Inverse matrix

Due to the type of operations involved, only a square matrix can have an inverse. To introduce these new concepts we will begin by making some considerations:

- Given any matrix $A \in \mathcal{M}_n$ we know that
$$A \cdot I_n = I_n \cdot A = A,$$
($\forall r \in \mathbb{R},\ 1 \cdot r = r \cdot 1 = r$). That is, I_n is the unit of $\mathcal{M}_n$.

- Given $a, b \in \mathbb{R},\ b \neq 0$, the division of a by b can be calculated as
$$\frac{a}{b} = a \cdot \frac{1}{b} = a \cdot b^{-1},$$
where b^{-1} is what is called the inverse of the number b.

- Given $b \in \mathbb{R},\ b \neq 0$, we know that its inverse is another real number that we write as b^{-1} and that is the only number that satisfies
$$b \cdot b^{-1} = b^{-1} \cdot b = 1.$$
The inverse of b is that number by which b must be multiplied to obtain 1.

- Not every real number has an inverse since it is not possible to calculate $0^{-1} = \frac{1}{0}$ because there is no number x such that
$$0 \cdot x = 1.$$

Due to all the above, it seems clear that, given a matrix $A \in \mathcal{M}_n$, if we want to define A^{-1}, we will have to find another matrix, $B \in \mathcal{M}_n$, such that

$$A \cdot B = B \cdot A = I_n$$

and then that matrix B will be the inverse of A, that is, $A^{-1} = B$.

Definition 30. Given $A \in \mathcal{M}_n$, if it exists, we call the inverse matrix of A and denote it by A^{-1} the unique matrix that satisfies:

$$A^{-1} \cdot A = A \cdot A^{-1} = I_n.$$

Examples 31.

1) Given $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ consider the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and then we have that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$
$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Therefore, according to the definition we have given, we have that:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

2) Given $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ if we take $\begin{pmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ we obtain:

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3,$$
$$\begin{pmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

and therefore $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$.

3) Given $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$, taking into account properties we know about the product of diagonal matrices, it is easy to calculate the inverse by taking the inverses of the elements of the main diagonal of B:

$$\begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{-5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3,$$
$$\begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{-5} \end{pmatrix} \cdot \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

and therefore $B^{-1} = \begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{-5} \end{pmatrix}$.

4) Let us determine whether $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has an inverse matrix. If A has an inverse, it will also be a square matrix of order 2 and therefore of the form

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we will have that

$$A \cdot A^{-1} = I_2 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_2 \Rightarrow \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} a+c=1 & b+d=1 \\ a+c=0 & b+d=0 \end{cases},$$

which is impossible since it is evident that the quantity $a+c$ cannot be simultaneously equal to 1 and equal to 0. We therefore deduce that the matrix A does not have an inverse matrix.

5) Let's try to calculate the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

To do this we will apply the same technique that we used in the previous example. In this way, if A has an inverse we know that it will also be a square matrix of order 2 and therefore must be of the form

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $A \cdot A^{-1} = I_2$ we will have that

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 3a+c & 3b+d \\ 2a+c & 2b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 3a+c=1, \ 3b+d=0, \\ 2a+c=0, \ 2b+d=1. \end{cases}$$

$$\overset{\text{Solving the system}}{\Rightarrow} \begin{cases} a=1, & b=-1, \\ c=-2, & d=3. \end{cases}$$

See that in the end we obtain a linear system with four equations and four unknowns that is easily solved so that finally we have calculated the inverse matrix which will be

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

Definition 32. We say that $A \in \mathcal{M}_n$ is a regular matrix if the inverse matrix of A exists and otherwise we say that A is a singular or non-regular matrix.

Example 33. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is singular, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ are regular.

Properties 34.

i) If $A, B \in \mathcal{M}_n$ *are regular then the matrix* $A \cdot B$ *is also regular and furthermore it is verified that:*

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.$$

ii) If $A \in \mathcal{M}_n$ *is regular then* A^t *is also regular and furthermore it is verified that:*

$$(A^t)^{-1} = (A^{-1})^t.$$

iii) Given the diagonal matrix $A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$ *such that* $a_1 \neq 0$*,* $a_2 \neq 0$*,...,*$a_n \neq 0$*, we have that* A *is regular and furthermore:*

$$A^{-1} = \begin{pmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_n} \end{pmatrix}.$$

iv) If $A \in \mathcal{M}_n$ *is regular, then* A^{-1} *is also regular and*

$$(A^{-1})^{-1} = A.$$

v) If $A \in \mathcal{M}_n$ is regular and we take $r \in \mathbb{R}$, $r \neq 0$, then $r \cdot A$ is regular and it is verified that:

$$(r \cdot A)^{-1} = \frac{1}{r} \cdot A^{-1}.$$

Examples 35.
1) Given $I_n \in \mathcal{M}_n$ we know that $I_n \cdot I_n = I_n$ so the identity matrix is regular and its inverse is itself:

$$(I_n)^{-1} = I_n.$$

2) Given $A, B, C \in \mathcal{M}_n$ we are going to calculate the inverse $(A \cdot B \cdot C)^{-1}$. To do this we will repeatedly use property *i)*.

$$(A \cdot B \cdot C)^{-1} = ((A \cdot B) \cdot C)^{-1} = (\text{property } i)) = C^{-1} \cdot (A \cdot B)^{-1}$$
$$= (\text{property } i)) = C^{-1} \cdot (B^{-1} \cdot A^{-1}) = C^{-1} \cdot B^{-1} \cdot A^{-1}.$$

In general,

$$(A_1 \cdot A_2 \cdot \dots \cdot A_k)^{-1} = A_k^{-1} \cdot \dots \cdot A_2^{-1} \cdot A_1^{-1}.$$

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3 Linear combinations, linear independence. Rank of a matrix

Example 36. We conduct the study in seven cities that we will call A, B, C, D, E, F and G. In each of them we will initially study two variables:

$$\begin{aligned} N_C &= \text{ Number of cars present in the city,} \\ N_M &= \text{ Number of motorcycles.} \end{aligned}$$

After the corresponding data collection we obtain the following values for these variables in each city (expressed in thousands of vehicles of each type):

	N_C	N_M
City A	7	6
City B	8	5
City C	10	5
City D	6	6
City E	4	5
City F	20	10
City G	9	5

We intend to analyze the recycling of tires and motor-derived waste. For this it is reasonable that we study in each city two new variables:

$$N_R = \text{ Number of tires circulating,}$$
$$N_m = \text{ Number of engines in use.}$$

We could perform new data collections in the cities of the study to obtain the information of these other two variables, however,

$$N_R = 4N_C + 2N_M \qquad \text{and} \qquad N_m = N_C + N_M. \qquad (1)$$

In this way we can calculate

	N_C	N_M	N_R	N_m
City A	7	6	40	13
City B	8	5	42	13
City C	10	5	50	15
City D	6	6	36	12
City E	4	5	26	9
City F	20	10	100	30
City G	9	5	46	14

Actually, each of the four variables is a 7-tuple and

$$\underbrace{\begin{pmatrix} 40 \\ 42 \\ 50 \\ 36 \\ 26 \\ 100 \\ 46 \end{pmatrix}}_{=N_R} = 4 \underbrace{\begin{pmatrix} 7 \\ 8 \\ 10 \\ 6 \\ 4 \\ 20 \\ 9 \end{pmatrix}}_{=N_C} +2 \underbrace{\begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ 5 \\ 10 \\ 5 \end{pmatrix}}_{=N_M} \qquad \text{and} \qquad \underbrace{\begin{pmatrix} 13 \\ 13 \\ 15 \\ 12 \\ 9 \\ 30 \\ 14 \end{pmatrix}}_{=N_m} = \underbrace{\begin{pmatrix} 7 \\ 8 \\ 10 \\ 6 \\ 4 \\ 20 \\ 9 \end{pmatrix}}_{=N_C} + \underbrace{\begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ 5 \\ 10 \\ 5 \end{pmatrix}}_{=N_M}$$

We can verify that the information of N_R and N_m depends on N_C and N_M and therefore it is not necessary for us to take data in each city for these variables, we simply have to obtain the information for N_R and N_m by combining what we already have in N_C and N_M.

Other variables that are obtained by combination of N_C and N_M could be:

- N_P = Maximum number of transportable passengers,
- N_F = Number of nighttime illumination headlights.

Evidently

$$N_P = 5N_C + 2N_M \qquad \text{and} \qquad N_F = 2N_C + N_M,$$

and using tuple calculus,

$$\underbrace{\begin{pmatrix} 47 \\ 50 \\ 60 \\ 42 \\ 30 \\ 120 \\ 55 \end{pmatrix}}_{=N_P} = 5 \underbrace{\begin{pmatrix} 7 \\ 8 \\ 10 \\ 6 \\ 4 \\ 20 \\ 9 \end{pmatrix}}_{=N_C} +2 \underbrace{\begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ 5 \\ 10 \\ 5 \end{pmatrix}}_{=N_M} \qquad \text{and} \qquad \underbrace{\begin{pmatrix} 20 \\ 21 \\ 25 \\ 18 \\ 13 \\ 50 \\ 23 \end{pmatrix}}_{=N_F} = 2 \underbrace{\begin{pmatrix} 7 \\ 8 \\ 10 \\ 6 \\ 4 \\ 20 \\ 9 \end{pmatrix}}_{=N_C} + \underbrace{\begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ 5 \\ 10 \\ 5 \end{pmatrix}}_{=N_M}$$

Actually any variable, N, that is obtained as a combination of N_C and N_M will be of the form

$$N = \alpha N_C + \beta N_M$$

and its values can be calculated by the tuple operation

$$N = \alpha \underbrace{\begin{pmatrix} 7 \\ 8 \\ 10 \\ 6 \\ 4 \\ 20 \\ 9 \end{pmatrix}}_{=N_C} + \beta \underbrace{\begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ 5 \\ 10 \\ 5 \end{pmatrix}}_{=N_M}.$$

For different values of α and β we can obtain an infinity of variables that are combinations of N_C and N_M but in all cases their information will be superfluous once we know these last two.

On the other hand, it seems clear that the number of cars in a city is completely independent of the number of motorcycles. Thus, the variables N_C and N_M are independent of each other and both are essential so we need to take the data of both without one being obtainable from the other. That is, there is no formula of the type

$$N_C = \alpha N_M \qquad \text{or} \qquad N_M = \alpha N_C$$

Thus, it seems clear that in this problem the essential variables are N_C and N_M from which we can derive others as combinations.

Definition 37. Consider the n-tuples $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$. Then:

i) We say that the n-tuple, $w \in \mathbb{R}^n$, is a linear combination of $v_1, v_2, \ldots, v_m$ if

$$w = \alpha_1 v_1 + \alpha_2 v2 + \cdots + \alpha_m v_m$$

for certain real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ that we call coefficients of the combination. The set of all linear combinations of $v_1, v_2, \ldots, v_m$ is denoted $\langle v_1, v_2, \ldots, v_m \rangle$.

ii) We say that $v_1, v_2, \ldots, v_m$ are linearly independent if none of them can be obtained as a linear combination of the others. We say that a single tuple is independent (i.e., $m = 1$) provided it is not null.

iii) We say that $v_1, v_2, \ldots, v_m$ are linearly dependent if they are not independent.

- The tuples N_R, N_m, N_P and N_F are linear combinations of N_C and N_M. In other words,

$$N_R, N_m, N_P, N_F \in \langle N_C, N_M \rangle.$$

  We can also obtain many other combinations of N_C and N_M and all of them will be of the form $\alpha N_C + \beta N_M$ for certain numbers $\alpha, \beta \in \mathbb{R}$. Therefore,

$$\langle N_C, N_M \rangle = \{\alpha N_C + \beta N_M : \alpha, \beta \in \mathbb{R}\}.$$

- The set formed by the tuples $N_C, N_M, N_R, N_m, N_P, N_F$ is linearly dependent.
- The tuples N_C and N_M are independent.

Dependent tuples are superfluous in the following sense:

Property 38. *Given the n-tuples w and $v_1, v_2, \ldots, v_m$, it holds that*

$$\underbrace{w \in \langle v_1, v_2, \ldots, v_m\rangle}_{\substack{\text{If } w \text{ is obtained as}\\ \text{combination of } v_1, v_2, \ldots, v_m\text{'}}} \Leftrightarrow \underbrace{\langle w, v_1, v_2, \ldots, v_m\rangle}_{\substack{\text{all combinations}\\ \text{obtained using } w,}} = \underbrace{\langle v_1, v_2, \ldots, v_m\rangle.}_{\substack{\text{can also be obtained}\\ \text{if we remove } w}}.$$

For example, combining N_C, N_M and N_R we can obtain

$$\langle N_C, N_M, N_R\rangle.$$

Now, N_R can be obtained as a combination of the other two. Consequently,

$$\underbrace{\langle N_C, N_M, N_R\rangle}_{\substack{\text{all combinations}\\ \text{obtained using } N_R,}} = \underbrace{\langle N_C, N_M\rangle.}_{\substack{\text{can also be obtained}\\ \text{if we remove } N_R}} \quad .$$

In other words,

$$\langle N_C, N_M, \cancel{N}_R\rangle.$$

Examples 39.

1) Consider the columns

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

If we add them multiplied by the numbers 5, 2 and -1 we obtain

$$5 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} =$$

$$= \boxed{\begin{pmatrix} 18 \\ 11 \\ -8 \end{pmatrix}}.$$

Therefore,

$$\begin{pmatrix} 18 \\ 11 \\ -8 \end{pmatrix} \in \langle \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \rangle.$$

2) Let us now take the 5-tuples

$$\begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix}.$$

Again we can combine them to obtain a row different from the initial ones. For example we can multiply by 3 and 2,

$$3\begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix} + 2\begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}6 & 9 & 0 & 0 & 18\end{pmatrix} + \begin{pmatrix}-2 & 4 & 6 & 0 & 2\end{pmatrix} = \boxed{\begin{pmatrix}4 & 13 & 6 & 0 & 20\end{pmatrix}}.$$

If we had chosen different coefficients

$$4\begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix} + (-1)\begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}8 & 12 & 0 & 0 & 24\end{pmatrix} + \begin{pmatrix}1 & -2 & -3 & 0 & -1\end{pmatrix} = \boxed{\begin{pmatrix}9 & 10 & -3 & 0 & 23\end{pmatrix}}.$$

In short we have that

$$\begin{pmatrix}4 & 13 & 6 & 0 & 20\end{pmatrix}, \begin{pmatrix}9 & 10 & -3 & 0 & 23\end{pmatrix} \in \left\langle \begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix}, \begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix} \right\rangle.$$

However,

$$\underbrace{a_1\begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix} + a_2\begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix}}_{\substack{= \begin{pmatrix}2a_1 - a_2 & 3a_1 + 2a_2 & 3a_2 & \boxed{0} & 6a_1 + a_2\end{pmatrix} \\ \text{the result always has} \\ \text{a 0 in the fourth position}}} \quad \text{and} \quad \underbrace{\begin{pmatrix}0 & 0 & 0 & \boxed{1} & 0\end{pmatrix}}_{\substack{\text{the row we intend to} \\ \text{obtain has a 1} \\ \text{in the fourth position}}}$$

That is,

$$\begin{pmatrix}0 & 0 & 0 & 1 & 0\end{pmatrix} \notin \left\langle \begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix}, \begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix} \right\rangle$$

and consequently

$$\left\langle \begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix}, \begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix} \right\rangle \neq \mathbb{R}^5.$$

That is, not every 5-tuple can be obtained as a linear combination of $\begin{pmatrix}2 & 3 & 0 & 0 & 6\end{pmatrix}$ and $\begin{pmatrix}-1 & 2 & 3 & 0 & 1\end{pmatrix}$.

3) Consider the columns $\begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}$. By performing linear combinations of these columns it is possible to obtain new ones. The set of all their linear combinations will be

$$\left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\rangle =$$

$$= \left\{ a_1 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} : a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}.$$

For example, taking $a_1 = 2$, $a_2 = -1$, $a_3 = 1$ and $a_4 = 3$ we obtain the linear combination

$$2 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \\ -11 \\ 1 \end{pmatrix}.$$

The question is whether it is possible to obtain the same linear combinations with fewer columns or, said in another way, if there is any of the four columns that is superfluous. We verify that

$$\begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}. \tag{2}$$

Directly if we apply **Property 38**, we know then that we can dispense with that third column.

For example, $\begin{pmatrix} 13 \\ -1 \\ -11 \\ 1 \end{pmatrix}$ is a combination of the four columns but using (2) we have that

$$\begin{pmatrix} 13 \\ -1 \\ -11 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + 2 \underbrace{\begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix}}_{\substack{\text{We substitute} \\ \text{using (2)}}} +3 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \Rightarrow$$

$$\downarrow$$

$$\overbrace{\left(2 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right)}$$

$$\Rightarrow \begin{pmatrix} 13 \\ -1 \\ -11 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Therefore, it is sufficient to use the first, second and fourth columns.

The fact that we can express the third column as a linear combination of the others has allowed us to eliminate it from the linear combination. Actually, this same argument is valid for any combination of the four columns and consequently

$$\underbrace{\left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \boxed{\begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix}}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\rangle}_{\substack{\text{All combinations obtained} \\ \text{using the four columns,}}} = \underbrace{\left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\rangle}_{\substack{\text{can also be obtained} \\ \text{if we remove the third}}}.$$

In short, the third column is superfluous when obtaining linear combinations and we could eliminate it,

$$\left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \cancel{\begin{pmatrix} 5 \\ -1 \\ -7 \\ -3 \end{pmatrix}}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\rangle.$$

Remark.

★ The zero tuple can always be obtained as a linear combination of any tuples $v_1, v_2, \ldots, v_m$ (that is, it always holds that $0 \in \langle v_1, v_2, \ldots, v_m \rangle$).

$$0 = 0v_1 + v_2 + \cdots + 0v_m.$$

This way of obtaining the zero tuple, being the simplest, is called 'trivial'.

Example 40. Given the rows $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & 3 & 9 \end{pmatrix}$, we can obtain the zero row (in this case the zero row will be $0_{1\times 3} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$) by taking equal to zero the two coefficients of the linear combination in the form

$$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} + 0 \begin{pmatrix} 4 & 3 & 9 \end{pmatrix}.$$

This would be the trivial way to obtain the zero tuple.

★ As a consequence of the previous comment, any set of tuples that contains the tuple 0 will always be linearly dependent. Indeed, we know that the tuple 0 can always be obtained as a linear combination of the others and therefore the set of tuples must be dependent.

Example 41. Without needing to perform any calculation we know that the tuples $(1, 2, -1), (2, 1, 1), (0, 0, 0)$ are dependent since one of them is the zero tuple that can be obtained as a linear combination of the others in the form

$$(0, 0, 0) = 0(1, 2, -1) + 0(2, 1, 1).$$

★ It is evident that if we have more tuples with them we will also be able to perform more linear combinations. That is, if we have the m tuples $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$ and additionally we take q more tuples, $w_1, w_2, \ldots, w_q \in \mathbb{R}^n$, it holds that

$$\underbrace{\langle v_1, v_2, \ldots, v_m \rangle}_{\substack{\text{All combinations}\\ \text{we can obtain}\\ \text{combining } v_1,\ldots,v_m}} \subseteq \underbrace{\langle v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_q \rangle}_{\substack{\text{we can also obtain them}\\ \text{combining } v_1,\ldots,v_m,w_1,\ldots,w_q}}.$$

Example 42. Given $(2,3,-1,0),(1,2,1,1) \in \mathbb{R}^4$ and additionally $(1,6,3,1),(0,3,0,1) \in \mathbb{R}^4$, any combination of the first two, for example

$$3(2,3,-1,0)+2(1,2,1,1) = (8,13,-1,2),$$

can also be written as a combination of the four, for example

$$(8,13,-1,2) = 3(2,3,-1,0)+2(1,2,1,1)+0(1,6,3,1)+0(0,3,0,1).$$

Therefore,

$$\langle (2,3,-1,0),(1,2,1,1) \rangle \subseteq \langle (2,3,-1,0),(1,2,1,1), (1,6,3,1),(0,3,0,1)\rangle.$$

★ If $v_1, v_2, \ldots, v_p$ are independent, any subset of tuples we choose from among them are also linearly independent.

Example 43. It is possible to verify that the 4-tuples

$$(1, 2, -1, 1), (2, 1, 1, 1), (0, -1, 1, 1), (2, -2, 1, 1)$$

are independent. In such a case any subset of them that we take will also be independent. For example,

$$(1, 2, -1, 1), (0, -1, 1, 1), (2, -2, 1, 1)$$

are independent tuples.

★ If among the tuples $v_1, v_2, \ldots, v_p$ any of them appears repeated, then said tuples are linearly dependent.

Example 44. The 4-tuples

$$(3,2,-1,2),\ (2,1,2,1),\ (3,2,-1,2),\ (7,2,3,1)$$

are dependent since one of them appears repeated.

$$\underbrace{(3,2,-1,2)}_{\text{The repeated tuple}} = \underbrace{0(2,1,2,1) + 1\underbrace{(3,2,-1,2)}_{\text{appears among the remaining ones}} + 0(7,2,3,1)}_{\text{and therefore we can obtain it easily as a linear combination.}}$$

Aquí está la traducción al inglés del fragmento sobre matrices, cumpliendo estrictamente con todas las indicaciones proporcionadas.

“‘latex

3.1 Basic techniques for studying linear dependence

Specifically, we need to solve the following problems:

a) Determine whether a given tuple can or cannot be obtained by combining others.

b) Determine whether a set of tuples are dependent or independent.

Let’s see how we can do this.

3.1.1 Determining if a tuple is a linear combination of others

We know that a tuple, $w \in \mathbb{R}^n$, is a linear combination of

$$v_1, v_2, \ldots, v_m \in \mathbb{R}^n$$

if we can find numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m.$$

In reality, when $\alpha_1, \alpha_2, \ldots, \alpha_m$ are unknown, the expression above constitutes a system of linear equations whose variables are the coefficients we want to determine.

Let's see this better in the following examples.

Examples 45.

1) Determine if the tuple

$$N_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \\ -1 \\ 10 \\ 4 \end{pmatrix}$$

is a linear combination of the tuples N_C and N_M from **Example 36**. N_1 will be a linear combination if we can find the numbers $\alpha, \beta \in \mathbb{R}$ such that

$$N_1 = \alpha N_C + \beta N_M.$$

If we substitute the value of each tuple and perform the

$$\begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \\ -1 \\ 10 \\ 4 \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 7 \\ 8 \\ 10 \\ 6 \\ 4 \\ 20 \\ 9 \end{pmatrix}}_{=N_C} + \beta \underbrace{\begin{pmatrix} 6 \\ 5 \\ 5 \\ 6 \\ 5 \\ 10 \\ 5 \end{pmatrix}}_{=N_M} \Rightarrow \begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \\ -1 \\ 10 \\ 4 \end{pmatrix} = \begin{pmatrix} 7\alpha + 6\beta \\ 8\alpha + 5\beta \\ 10\alpha + 5\beta \\ 6\alpha + 6\beta \\ 4\alpha + 5\beta \\ 20\alpha + 10\beta \\ 9\alpha + 5\beta \end{pmatrix}$$

and if we now equate row by row we arrive at

$$\begin{cases} 7\alpha + 6\beta = 1 \\ 8\alpha + 5\beta = 3 \\ 10\alpha + 5\beta = 5 \\ 6\alpha + 6\beta = 0 \\ 4\alpha + 5\beta = -1 \\ 20\alpha + 10\beta = 10 \\ 9\alpha + 5\beta = 4 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \end{cases}.$$

Therefore, we have calculated the coefficients α and β that we need and we now know that N_1 can be written as

$$N_1 = N_C - N_M.$$

Thus $N \in \langle N_C, N_M \rangle$.

Note that the problem reduces to finding the solution of a system of linear equations.

Let us now study the same problem for the tuple

$$N_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Repeating the same steps, we must again find the coefficients α and β such that

$$N_2 = \alpha N_C + \beta N_M \Rightarrow$$

$$\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\end{pmatrix} = \alpha \underbrace{\begin{pmatrix}7\\8\\10\\6\\4\\20\\9\end{pmatrix}}_{=N_C} + \beta \underbrace{\begin{pmatrix}6\\5\\5\\6\\5\\10\\5\end{pmatrix}}_{=N_M} \Rightarrow \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}7\alpha+6\beta\\8\alpha+5\beta\\10\alpha+5\beta\\6\alpha+6\beta\\4\alpha+5\beta\\20\alpha+10\beta\\9\alpha+5\beta\end{pmatrix}$$

$$\Rightarrow \begin{cases} 7\alpha+6\beta=1\\ 8\alpha+5\beta=1\\ 10\alpha+5\beta=1\\ 6\alpha+6\beta=1\\ 4\alpha+5\beta=1\\ 20\alpha+10\beta=1\\ 9\alpha+5\beta=1 \end{cases}.$$

Inconsistent system. Therefore, we finally have that

$$N_2 \notin \langle N_C, N_M \rangle$$

and the tuple N_2 cannot be obtained by combining N_C and N_M.

2) Check if $(3,3,1) \in \langle (1,2,1),(1,1,1),(2,1,1) \rangle$.

Again we must find the coefficients, in this case three, needed to form the combination that produces the tuple $(3,3,1)$,

$$(3,3,1) = \alpha(1,2,1) + \beta(1,1,1) + \gamma(2,1,1).$$

Performing the operations and equating we have

$$(3,3,1) = (\alpha+\beta+2\gamma, 2\alpha+\beta+\gamma, \alpha+\beta+\gamma) \Rightarrow \begin{cases} \alpha+\beta+2\gamma = 3 \\ 2\alpha+\beta+\gamma = 3 \\ \alpha+\beta+\gamma = 1 \end{cases}.$$

To solve the system,

$$\begin{cases} \text{subtracting the last equation from the first} & \Rightarrow \ \gamma = 2, \\ \text{subtracting the last equation from the second} & \Rightarrow \ \alpha = 2, \\ \text{substituting and solving in the third} & \Rightarrow \ \beta = -3. \end{cases}$$

Therefore

$$(3,3,1) = 2(1,2,1) - 3(1,1,1) + 2(2,1,1)$$

and $(3,3,1) \in \langle (1,2,1),(1,1,1),(2,1,1) \rangle$.

3) Study whether $(1,0,0) \in \langle (1,2,1),(1,1,1),(2,3,2) \rangle$.

Repeating the process we will again obtain a system of linear equations,

$$(1,0,0) = \alpha(1,2,1)+\beta(1,1,1)+\gamma(2,3,2) \Rightarrow \begin{cases} \alpha+\beta+2\gamma = 1 \\ 2\alpha+\beta+3\gamma = 0 \\ \alpha+\beta+2\gamma = 0 \end{cases}$$

and it is evident that the first and the last equation cannot be satisfied at the same time since $\alpha+\beta+2\gamma$ cannot simultaneously be 1 and 0. Consequently, this system has no solution and we cannot find the coefficients α, β and γ. Therefore,

$$(1,0,0) \notin \langle (1,2,1),(1,1,1),(2,3,2) \rangle.$$

3.1.2 Study of linear dependence and independence

Given several tuples $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$ we know that it is always

$$0v_1 + 0v_2 + \cdots + 0v_m = 0.$$

But, is this the only way to obtain the tuple 0 as a combination of $v_1, v_2, \ldots, v_m$ or will there be others?

To justify this last point, let's return to **Example 36**. Since

$$N_R = 4N_C + 2N_M,$$

we know that these three tuples, $\{N_C, N_M, N_R\}$, are dependent. If we move all the terms in this last equality to the same side we have

$$4N_C + 2N_M - N_R = 0,$$

where $0 = (0, 0, 0, 0, 0, 0, 0)$ is the 7-tuple zero. In this way, since they are dependent, we can find a way to obtain the tuple 0 different from the trivial one.

Property 46. *Consider the tuples* $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$*. Then if the only values of the numbers* $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ *for which we obtain*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

are $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$*, said tuples are independent. Otherwise, the tuples will be dependent.*

Examples 47.

1) Let us study whether the tuples $(2,3,1),(4,6,4),(4,6,3) \in \mathbb{R}^3$ are dependent or independent. To do this

$$\alpha_1(2,3,1) + \alpha_2(4,6,4) + \alpha_3(4,6,3) = (0,0,0).$$

First we will perform the matrix operations indicated in this equality, which leads us to

$$(2\alpha_1 + 4\alpha_2 + 4\alpha_3, 3\alpha_1 + 6\alpha_2 + 6\alpha_3, \alpha_1 + 4\alpha_2 + 3\alpha_3) = (0,0,0)$$

and now equating both members we finally obtain the system of linear equations with three variables and three equations,

$$\left\{\begin{array}{l} 2\alpha_1 + 4\alpha_2 + 4\alpha_3 = 0 \\ 3\alpha_1 + 6\alpha_2 + 6\alpha_3 = 0 \\ \alpha_1 + 4\alpha_2 + 3\alpha_3 = 0 \end{array}\right. .$$

Solving this system we must determine the possible values for α_1, α_2 and α_3. In this case,

$$\left\{\begin{array}{lcl} 2\alpha_1 + 4\alpha_2 + 4\alpha_3 = 0 & \xrightarrow[\text{dividing by 2}]{} & \alpha_1 + 2\alpha_2 + \cancel{2\alpha_3 = 0} \\ 3\alpha_1 + 6\alpha_2 + 6\alpha_3 = 0 & \xrightarrow[\text{dividing by 3}]{} & \alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \\ \alpha_1 + 4\alpha_2 + 3\alpha_3 = 0 & & \alpha_1 + 4\alpha_2 + 3\alpha_3 = 0 \end{array}\right. ,$$

Now the system becomes

$$\begin{cases} \alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 & \xrightarrow[\text{equation 2 minus equation 1}]{} \alpha_3 = -2\alpha_2 \\ \alpha_1 + 4\alpha_2 + 3\alpha_3 = 0 & \xrightarrow[\text{substituting}]{} \alpha_1 = 2\alpha_2 \end{cases}.$$

Since we have infinitely many solutions, these tuples are dependent.

If we take $\alpha_2 = 1$ we obtain,

$$\alpha_1 = 2,\ \alpha_2 = 1,\ \alpha_3 = -2,$$

which leads us to the following non-trivial expression of the zero tuple,

$$2(2,3,1) + (4,6,4) - 2(4,6,3) = (0,0,0).$$

The existence of an alternative to the trivial form implies dependence,

$$(0,0,0) = \boxed{2}\,(2,3,1) + 1(4,6,4) + (-2)(4,6,3)$$

$$\Rightarrow (0,0,0) - \boxed{2}\,(2,3,1) = 1(4,6,4) + (-2)(4,6,3)$$

$$\Rightarrow (2,3,1) = \frac{1}{-\boxed{2}}(4,6,4) + \frac{2}{-\boxed{2}}(4,6,3)$$

We see in this way that $(2,3,1)$ is a linear combination of $(4,6,4)$ and $(4,6,3)$ and we have again the dependence.

2) Let us check if the tuples $(1,1,3), (2,0,-1), (-1,0,1)$ are dependent or independent.

To do this, we will use **Property 2**.

$$a_1(1,1,3) + a_2(2,0,-1) + a_3(-1,0,1) = (0,0,0).$$

$$a_1(1,1,3) + a_2(2,0,-1) + a_3(-1,0,1) = (0,0,0)$$
$$\Downarrow$$
$$(a_1, a_1, 3a_1) + (2a_2, 0, -a_2) + (-a_3, 0, a_3) = (0,0,0)$$
$$\Downarrow$$
$$(a_1 + 2a_2 - a_3, a_1, 3a_1 - a_2 + a_3) = (0,0,0)$$

$\Downarrow$ For two tuples to be equal, each of their components must be equal.

$$\begin{cases} a_1 + 2a_2 - a_3 = 0 \\ a_1 = 0 \\ 3a_1 - a_2 + a_3 = 0 \end{cases}$$

The coefficients a_1, a_2 and a_3 must satisfy the equations of the previous system. Now, if we solve the system we easily obtain that

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0$$

3) Study the dependence and independence of the tuples $(2,0,0,1)$, $(0,1,0,-1)$, $(4,-1,0,3)$, $(0,1,1,0)$.

We will use the same technique as in the previous section:

$$x(2,0,0,1)+y(0,1,0,-1)+z(4,-1,0,3)+w(0,1,1,0)=(0,0,0,0)$$

$$\Downarrow$$

$$(2x,0,0,x)+(0,y,0,-y)+(4z,-z,0,3z)+(0,w,w,0)=(0,0,0,0)$$

$$\Downarrow$$

$$(2x+4z,y-z+w,w,x-y+3z)=(0,0,0,0)$$

$$\Downarrow$$

$$\begin{cases} 2x+4z=0 \\ y-z+w=0 \\ w=0 \\ x-y+3z=0 \end{cases}.$$

If we solve the system we observe that we can only solve for three of the variables in the form

$$\begin{cases} x=-2z \\ y=z \\ w=0 \end{cases}.$$

It has infinitely many solutions. They are dependent, we deduce that the tuples are dependent.

4) Consider the n elements $e_1, e_2, e_3, \ldots, e_n$ of $\mathbb{R}^n$ defined as follows:

$$\begin{aligned} e_1 &= (1,0,0,0,\ldots,0,0), \\ e_2 &= (0,1,0,0,\ldots,0,0), \\ e_3 &= (0,0,1,0,\ldots,0,0), \\ &\vdots \\ e_n &= (0,0,0,0,\ldots,0,1). \end{aligned}$$

To do this, suppose we obtain the tuple 0 of $\mathbb{R}^n$ as a linear combination of e_1, e_2, $\ldots$, e_n in the form

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$$

and let's see that in such a case the only possibility is that all the coefficients α_1, α_2, $\ldots$, α_n are equal to zero.

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0 \Rightarrow$$

$$\alpha_1(1,0,0,\ldots,0) + \alpha_2(0,1,0,\ldots,0) + \ldots \alpha_n(0,0,0,\ldots,1) = (0,\ldots,0)$$

$$\Rightarrow (\alpha_1,0,0,\ldots,0) + (0,\alpha_2,0,\ldots,0) + (0,0,0,\ldots,\alpha_n) = (0,0,0,\ldots,0)$$

$$\Rightarrow (\alpha_1,\alpha_2,\ldots,\alpha_n) = (0,0,\ldots,0)$$

$$\Rightarrow \begin{cases} \alpha_1 = 0, \\ \alpha_2 = 0, \\ \quad \vdots \\ \alpha_n = 0. \end{cases}$$

Therefore all the coefficients α_1, α_2, ..., α_n are necessarily zero and as a consequence $e_1, e_2, \ldots, e_n$ are independent.

The tuples e_1, e_2,..., e_n of $\mathbb{R}_n$ are called coordinate n-tuples of $\mathbb{R}^n$. Let's look at some examples:

- In $\mathbb{R}^2$ the coordinate 2-tuples are 2:

$$e_1 = (1,0) \quad \text{and} \quad e_2 = (0,1).$$

  We also know that e_1 and e_2 are independent of each other.

- In $\mathbb{R}^3$ the coordinate 3-tuples will be 3:

$$e_1 = (1,0,0),\ e_2 = (0,1,0) \quad \text{and} \quad e_3 = (0,0,1).$$

  These three tuples are independent of each other.

- In $\mathbb{R}^4$ the 4 coordinate tuples are

$$e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),$$
$$\text{and} \quad e_4 = (0,0,0,1).$$

3.2 Rank of a Matrix

Definition 48. Given the matrix $A = (v_1|v_2|\dots|v_n) \in \mathcal{M}_{m\times n}$ whose column tuples are v_1, v_2, $\dots$, v_n, we call the rank of the matrix A, and denote it by

$$\mathrm{rango}(A) \quad \text{or} \quad \mathrm{r}(A),$$

the size of the largest subset of independent tuples that we can find among the column tuples v_1, v_2, $\dots$, v_n. By definition, we will say that the rank of the matrix $0_{m\times n}$ is 0.

Examples 49.

1) Let's take

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its column tuples are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, taking the first two columns of A, we have the set

$$\{e_1, e_2\}$$

which is independent and has size 2. We can ask ourselves if there will exist some set of independent columns with more elements. However, we know that any set that contains the tuple 0 will always be dependent. Therefore the last two columns can never be part of any independent set. From this we deduce that any set of column tuples larger than $\{e_1, e_2\}$ should include at least one of the tuples 0 and would not be independent. In this way, the largest set of column tuples independent that we can achieve is $\{e_1, e_2\}$ which has size 2 and consequently

$$\text{rango}(A) = 2.$$

2) Let's take the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Its column tuples,

$$(1, 0, 0, 0),\ (0, 1, 0, 0),\ (0, 0, 1, 0)$$

The largest linearly independent set that we can obtain from these three columns will therefore consist of taking them all and for that reason it will have size 3. Consequently,

$$\text{rango}(B) = 3.$$

3) Using the same arguments:

- rango $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 3.$

- rango $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4.$

- rango $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1.$

- rango $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 2.$

4) In **Definition 4** it is established that the rank of the zero matrix is always 0. Some examples of this are:

- $\text{rango}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$
- $\text{rango}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$

In reality, the column tuples of the zero matrix are all of them equal to the corresponding zero tuple and therefore we can never form any independent set with them. Hence the rank of these matrices is established as 0.

Property 50. *Given* $r, n, m \in \mathbb{N}$*:*

i) $\text{rango}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) = r.$

ii) $\text{rango}(I_r) = r.$

iii) $\text{rango}(0_{n\times m}) = 0.$

In other words, to calculate the rank of a matrix with all its elements zero except for ones on the diagonal, we only need to count the number of ones that appear in it. For example, the identity matrix of order r will always have rank r since it has r ones arranged diagonally.

The question is that so far we only know how to calculate the rank of matrices of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$. If we have any matrix we can try to transform it into one of this type.

Example 51. The matrix $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is not of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$, however it is sufficient to modify the order of the columns to obtain:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{Reordering columns}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

But the transformed matrix is of the type $A = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$ and its rank is 3 (three ones on the diagonal). As a consequence

$$\mathrm{r}(A) = 3.$$

We can ask ourselves the following questions

a) Are there more transformations like this one that modify the matrix but not the value of the rank?

b) Assuming the answer to question a) is affirmative: Will the available transformations allow us to convert any matrix into one of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$?

We will see that in both cases we have an affirmative answer.

Property 52. *Given the matrix $A \in \mathcal{M}_{m\times n}$, it holds that:*

i) If we modify the order of the rows or columns of A, the resulting matrix has the same rank as A.

ii) If we multiply one of the rows or columns of A by a number different from zero, the resulting matrix has the same rank as A.

iii) If we add to one column (respectively row) another column (respectively row) multiplied by a number, the resulting matrix has the same rank as A.

Definition 53. Given a matrix $A = (a_{ij})_{m\times n} \in \mathcal{M}_{m\times n}$ we call an elementary operation on A any of the following transformations:

- Multiply a row or column by a non-zero number.
- Modify the order of the rows or columns.
- Add to one column (respectively row) another column (resp. row) multiplied by any number.

In what follows we will use the following nomenclature to describe the elementary operations we perform on a matrix:

a) When we multiply the i-th column by a number k we indicate it by "kCi".

b) When we interchange column i with column j we indicate it by "C$i$$\leftrightarrow$C$j$".

c) When we add to column i the column j multiplied by a number k we denote it by "Ci=Ci+kCj".

d) The same operations for rows are denoted using the letter "F" instead of "C".

Examples 54.

1) Let's apply a series of elementary operations to the following matrix:

$$
\begin{aligned}
\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \end{pmatrix} \quad \xrightarrow[\text{C1}\leftrightarrow\text{C2}]{} \quad & \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 0 \end{pmatrix} \\
\xrightarrow[\text{F1=F1+2F2}]{} \; & \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \end{pmatrix} \\
\xrightarrow[\text{F1}\leftrightarrow\text{F2}]{} \quad & \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \\
\xrightarrow[\text{3C1}]{} \qquad & \begin{pmatrix} 3 & 6 & 9 \\ 0 & -1 & 0 \end{pmatrix}
\end{aligned}
$$

The matrices that appear on the right, all of them, have been obtained from $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \end{pmatrix}$ by applying elementary operations.

$$
\begin{aligned}
\text{rango} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \end{pmatrix} &= \text{rango} \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 0 \end{pmatrix} = \text{rango} \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \end{pmatrix} \\
&= \text{rango} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} = \text{rango} \begin{pmatrix} 3 & 6 & 9 \\ 0 & -1 & 0 \end{pmatrix}.
\end{aligned}
$$

2) To calculate the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -4 & 1 & 0 \end{pmatrix}$$

we can apply elementary operations to it to try to transform it into a matrix of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$. Let's see how we can do it:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -4 & 1 & 0 \end{pmatrix} \xrightarrow[\text{C2}\leftrightarrow\text{C3}]{} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \xrightarrow[\text{F2=F2-2F1}]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{F3=F3+4F1}]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In short, we calculate the rank of A as

$$\text{rango}(A) = \text{rango} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3.$$

The last example reproduces the method we intend to apply to calculate the rank of a matrix. Given the matrix A we will apply elementary operations to it to try to transform it into a matrix of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$,

$$A \xrightarrow[\text{Elementary operations}]{} \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Then,

$$\text{rango}(A) = \text{rango}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) = r.$$

The following question remains pending:

b) Will the available transformations allow us to convert any matrix into one of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$?

3.2.1 The Gaussian elimination method

The Gaussian elimination method or Gaussian elimination allows us to reduce, through elementary operations, any matrix to a matrix with ones on the diagonal and the rest of the elements zero. It is what is called an iterative method. That is, it is based on the repeated application of the same steps. These steps are the six we will see below and they will be the ones we will always apply to calculate the rank of a matrix.

We will see the steps to follow while reproducing them on a concrete example. So, these are the steps of the Gaussian elimination method:

1) We select in the matrix a row or column in which there exist at least two non-zero elements.
In general we will select the row or column with a larger number of zeros.

2) In the row or column selected in the previous section we choose a non-zero element which we call the 'pivot':

- In the case of manual calculations, the choice of the pivot element with value equal to 1 or -1 can simplify the operations.

- In the case of precise calculations, the best results are obtained by selecting as the pivot the element of the row or column with the largest absolute value.

Example 55.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 2 & 1 \\ \mathbf{2} & \mathbf{0} & \underline{\overline{\mathbf{1}}} & \mathbf{0} \\ 1 & -1 & 2 & 3 \end{pmatrix}.$$

---

3) We use the pivot to cancel all the elements of the row or column initially selected.

Example 56.

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 2 & 1 \\ \mathbf{2} & \mathbf{0} & \underline{\overline{\mathbf{1}}} & \mathbf{0} \\ 1 & -1 & 2 & 3 \end{pmatrix} \xrightarrow[\text{C1=C1-2C3}]{} \begin{pmatrix} 1 & 2 & 0 & 1 \\ -3 & -1 & 2 & 1 \\ \mathbf{0} & \mathbf{0} & \underline{\overline{\mathbf{1}}} & \mathbf{0} \\ -3 & -1 & 2 & 3 \end{pmatrix}.$$

---

4) We use the pivot to cancel the elements of the row or column perpendicular to the one we had selected in step 1 that intersects at the height of the pivot.

Example 57.

$$\begin{pmatrix} 1 & 2 & \mathbf{0} & 1 \\ -3 & -1 & \mathbf{2} & 1 \\ \mathbf{0} & \mathbf{0} & \underline{\overline{\mathbf{1}}} & \mathbf{0} \\ -3 & -1 & \mathbf{2} & 3 \end{pmatrix} \xrightarrow[\substack{\text{F2=F2-2F3} \\ \text{F4=F4-2F3}}]{} \begin{pmatrix} 1 & 2 & \mathbf{0} & 1 \\ -3 & -1 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \underline{\overline{\mathbf{1}}} & \mathbf{0} \\ -3 & -1 & \mathbf{0} & 3 \end{pmatrix}$$

---

5) Whenever there remains some row or column with more than one non-zero element we return to step 1.

Example 58.

$$\begin{pmatrix} 1 & \mathbf{2} & 0 & 1 \\ -3 & \overline{\underline{\mathbf{-1}}} & 0 & 1 \\ 0 & \mathbf{0} & 1 & 0 \\ -3 & \mathbf{-1} & 0 & 3 \end{pmatrix} \xrightarrow[\substack{\text{F1=F1+2F2} \\ \text{F4=F4-F2}}]{}$$

$$\begin{pmatrix} -5 & \mathbf{0} & 0 & 3 \\ -3 & \overline{\underline{\mathbf{-1}}} & 0 & 1 \\ 0 & \mathbf{0} & 1 & 0 \\ 0 & \mathbf{0} & 0 & 2 \end{pmatrix} \xrightarrow[\substack{\text{C1=C1-3C2} \\ \text{C4=C4+C2}}]{} \begin{pmatrix} -5 & \mathbf{0} & 0 & 3 \\ \mathbf{0} & \overline{\underline{\mathbf{-1}}} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & 1 & 0 \\ 0 & \mathbf{0} & 0 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} -5 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\underline{\mathbf{2}}} \end{pmatrix} \to \begin{pmatrix} -5 & 0 & 0 & \mathbf{3} \\ 0 & -1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\underline{\mathbf{2}}} \end{pmatrix} \xrightarrow[\text{F1=F1-}\frac{3}{2}\text{F4}]{} \begin{pmatrix} -5 & 0 & 0 & \mathbf{0} \\ 0 & -1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\underline{\mathbf{2}}} \end{pmatrix}.$$

6) If necessary, the rows or columns are reordered to bring the matrix to diagonal form. If on the main diagonal appear non-zero elements different from 1 we can appropriately divide the corresponding row or column to transform them into 1.

Example 59.

$$\begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow[\substack{\mathrm{F1}=\frac{1}{-5}\mathrm{F1} \\ \mathrm{F2}=\frac{1}{-1}\mathrm{F2}}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4.$$

Finally we have obtained a matrix of the type $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$. We can now calculate the rank of the initial matrix A:

$$\text{rango} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 2 & 3 \end{pmatrix} = \text{rango} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4.$$

In this way, the idea for calculating the rank of a matrix A is based on the following steps:

1. We apply the Gaussian elimination method to find a list of elementary operations, let's call it L, that transforms A as follows:
$$A \xrightarrow[\text{Operations } L]{} \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$
2. We use **Property 8** which guarantees us that
$$\text{rango}(A) = \text{rango}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$
3. Finally, we apply **Property 6** to perform the calculation:
$$\text{rango}(A) = \text{rango}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) = r.$$

3.2.2 Row rank and other basic properties

Theorem 60. *Given a matrix $A \in \mathcal{M}_{m\times n}$ it holds that:*

i) $\text{rango}(A) = \text{rango}(A^t)$.

ii) The rank of A is the size of the largest independent set formed by row tuples of the matrix A.

An immediate consequence of the last theorem is the fact that the rank of a matrix will be less than or equal to both the number of columns and the number of rows of the matrix. This is reflected more precisely in the following corollary.

Corolario 1. *Given $A \in \mathcal{M}_{m\times n}$ it holds that*

$$\text{rango}(A) \le m \quad \textit{and} \quad \text{rango}(A) \le n.$$

3.2.3 Calculations with dependence, independence, linear combinations using the rank

Property 61. *Consider the n-tuples*

$$v_1, v_2, \ldots, v_m \quad \textit{and} \quad w$$

i) $v_1, v_2, \ldots, v_m$ *are independent*

$$\Leftrightarrow \text{rango}(v_1|v_2|\cdots|v_m) = m.$$

ii) $w \in \langle v_1, v_2, \ldots, v_m \rangle$

$$\Leftrightarrow \text{rango}(v_1|v_2|\cdots|v_m) = \text{rango}(v_1|v_2|\cdots|v_m|w).$$

- **What happens when they are dependent?**

If $v_1, v_2, \ldots, v_m$ are dependent, how do we discard the superfluous tuples? We will have that

$$\text{rango}(v_1|v_2|\cdots|v_m) = r < m$$

and there will be r of them that are independent. We know then that for many of the operations related to linear combinations we can choose those independent tuples and discard the others. But, how do we find out which of the tuples are independent and which are discardable?

Examples 62.

1) Check if the tuples $v_1 = (2,1,3,-1,2)$, $v_2 = (4,2,6,-2,4)$, $v_3 = (-2,-1,0,-5,1)$, $v_4 = (2,1,2,1,1)$ and $v_5 = (2,1,5,-5,4)$ are independent.

We will solve the exercise by applying **Property 18**. To do this, we will begin by calculating the rank of the matrix obtained by putting the tuples in question into columns,

$$\text{rango}(v_1|v_2|v_3|v_4|v_5) = \text{rango}\begin{pmatrix} 2 & 4 & -2 & 2 & 2 \\ 1 & 2 & -1 & 1 & 1 \\ 3 & 6 & 0 & 2 & 5 \\ -1 & -2 & -5 & 1 & -5 \\ 2 & 4 & 1 & 1 & 4 \end{pmatrix} = 2.$$

Therefore they are not independent.

2) The tuples from the previous section can be combined to obtain others. Are the five given tuples necessary to obtain all those combinations or, on the contrary, are there superfluous tuples?

If we take the first two columns we have

$$\operatorname{rango}(v_1|v_2) = \operatorname{rango}\begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 6 \\ -1 & -2 \\ 2 & 4 \end{pmatrix} = 1$$

and therefore they are not independent. Let's take instead the first and third columns,

$$\operatorname{rango}(v_1|v_3) = \operatorname{rango}\begin{pmatrix} 2 & -2 \\ 1 & -1 \\ 3 & 0 \\ -1 & -5 \\ 2 & 1 \end{pmatrix} = 2$$

and these two columns are independent. Therefore,

$$\langle v_1, \cancel{v}_2, v_3, \cancel{v}_4, \cancel{v}_5 \rangle = \langle (2,1,3,-1,2), (-2,-1,0,-5,1) \rangle.$$

We could also have chosen, for example, the fourth and the fifth

$$\text{rango}(v_4|v_5) = \text{rango}\begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 2 & 5 \\ 1 & -1 \\ 1 & 4 \end{pmatrix} = 2.$$

In this way, we have two answers for the same exercise. Combining the tuples

$$(2,1,3,-1,2) \quad \text{and} \quad (-2,-1,0,-5,1)$$

we obtain the same combinations as with the five initial tuples and, at the same time, combining

$$(2,1,2,1,1) \quad \text{and} \quad (2,1,5,-5,4)$$

we also obtain all those combinations.

3) Check if $w_1 = (1,2,3,2,1)$ and $w_2 = (0,0,3,-6,3)$ can be obtained as a linear combination of the tuples v_1, v_2, v_3, v_4, v_5 from section **1)**.

Let's start with $w_1 = (1,2,3,2,1)$. Applying part *ii)* of **Property 18** we know that

$$w_1 \in \langle v_1, v_2, v_3, v_4, v_5 \rangle$$

$$\Leftrightarrow \text{rango}(v_1|v_2|v_3|v_4|v_5) = \text{rango}(v_1|v_2|v_3|v_4|v_5|w_1)$$

That is, we must check if the equality is true

$$¿\,\underbrace{\text{rango}\begin{pmatrix} 2 & 4 & -2 & 2 & 2 \\ 1 & 2 & -1 & 1 & 1 \\ 3 & 6 & 0 & 2 & 5 \\ -1 & -2 & -5 & 1 & -5 \\ 2 & 4 & 1 & 1 & 4 \end{pmatrix}}_{=2 \text{ (section } \mathbf{1)})} = \underbrace{\text{rango}\begin{pmatrix} 2 & 4 & -2 & 2 & 2 & 1 \\ 1 & 2 & -1 & 1 & 1 & 2 \\ 3 & 6 & 0 & 2 & 5 & 3 \\ -1 & -2 & -5 & 1 & -5 & 2 \\ 2 & 4 & 1 & 1 & 4 & 1 \end{pmatrix}}_{=3}?$$

Therefore, $w_1 \notin \langle v_1, v_2, v_3, v_4, v_5 \rangle$.

We could have simplified these calculations if we had used the results from section **2)** since then we saw that

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle = \langle v_1, v_3 \rangle.$$

For example, for $w_2 = (0, 0, 3, -6, 3)$,

$$w_2 \in \langle v_1, v_3 \rangle \Leftrightarrow \text{rango}(v_1|v_3) = \text{rango}(v_1|v_3|w_2).$$

If we calculate the ranks of $(v_1|v_3)$ and $(v_1|v_3|w_2)$ we have

$$\underbrace{\text{rango}\begin{pmatrix} 2 & -2 \\ 1 & -1 \\ 3 & 0 \\ -1 & -5 \\ 2 & 1 \end{pmatrix}}_{=2 \text{ (section } \mathbf{1)})} = \underbrace{\text{rango}\begin{pmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & 3 \\ -1 & -5 & -6 \\ 2 & 1 & 3 \end{pmatrix}}_{=2}.$$

so yes it is a combination.

3.2.4 Column labeling

The scheme of elementary operations that makes it possible to calculate the rank of a matrix also allows us to detect the independent tuples. To do this it is enough to label each column with an indicator of the tuple to which it corresponds. After applying the scheme of elementary operations, the tuples whose labels appear over the ones on the diagonal of the reduced form will be directly independent.

Example 63. Consider the tuples $v_1 = (1,2,0,-1)$, $v_2 = (2,4,0,-2)$, $v_3 = (2,1,1,0)$ and $v_4 = (4,5,1,-2)$. To see how many and which of them are independent, we will put them in columns and calculate the rank of the resulting matrix. However, we will also label each column:

$$\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & 1 & 5 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & 0 & -2 \end{pmatrix} \end{array} \xrightarrow[\text{C4=C4-C3}]{} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & -2 \end{pmatrix} \end{array}$$

$$\xrightarrow[\substack{\text{F1=F1-2F3} \\ \text{F2=F2-F3}}]{} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & -2 \end{pmatrix} \end{array} \xrightarrow[\substack{\text{F2=F2-2F1} \\ \text{F4=F4+F1}}]{} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\xrightarrow[\substack{\text{C2=C2-2C1} \\ \text{C4=C4-2C1}}]{} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \xrightarrow[\substack{\text{C2}\leftrightarrow\text{ C3} \\ \text{F2}\leftrightarrow\text{ F3}}]{} \begin{array}{c} \begin{matrix} v_1 & v_3 & v_2 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

3.2.5 Obtaining all possible tuples

Given a certain set of n-tuples, $v_1, v_2, \ldots, v_m$, in some occasions it is important to determine if they are sufficient to generate via combinations any other n-tuple

The set of all n-tuples is $\mathbb{R}^n$, so the question we are posing here is whether with certain tuples $v_1, v_2, \ldots, v_m$ we can obtain all of $\mathbb{R}^n$, that is, whether

$$\langle v_1, v_2, \ldots, v_m \rangle = \mathbb{R}^n.$$

Property 64.

i) Given the n-tuples $v_1, v_2, \ldots, v_m$,

$$\langle v_1, v_2, \ldots, v_m \rangle = \mathbb{R}^n \Leftrightarrow \mathrm{rango}(v_1|v_2|\cdots|v_m) = n.$$

ii) The n-tuple coordinates of $\mathbb{R}^n$, $e_1, e_2, \ldots, e_n$ *satisfy*

$$\langle e_1, e_2, \ldots, e_n \rangle = \mathbb{R}^n.$$

iii) Given the n-tuples $v_1, v_2, \ldots, v_n$,

$$\langle v_1, v_2, \ldots, v_n \rangle = \mathbb{R}^n \Leftrightarrow \{v_1, v_2, \ldots, v_n\} \text{ is independent.}$$

iv) More than n, n-tuples cannot be independent.

v) With fewer than n, n-tuples it is not possible to obtain all of $\mathbb{R}^n$ *(i.e., all n-tuples).*

Examples 65.

1) Given the 4-tuples,

$$(1, 2, -1, 1),\ (2, 1, 0, -1),\ (0, 0, 1, -1) \text{ and } (1, 2, 0, 1),$$

we ask whether by combining them it is possible to obtain any other 4-tuple we desire.

Applying part *i)* of **Property 21** we know that

$$\langle (1, 2, -1, 1), (2, 1, 0, -1), (0, 0, 1, -1), (1, 2, 0, 1) \rangle = \mathbb{R}^4$$

will hold if, placed in columns, said tuples give rank 4. We have that

$$\text{rango} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix} = 4$$

so indeed any tuple of $\mathbb{R}^4$ can be obtained by combining those four.

2) Let's study the same problem now referring to the tuples $(1,2,3)$, $(2,1,3)$, $(-1,2,1)$ and $(2,3,5)$. We then ask if by combining them it is possible to obtain any 3-tuple, that is, if

$$\langle (1,2,3),(2,1,3),(-1,2,1),(2,3,5)\rangle = \mathbb{R}^3. \tag{3}$$

Without needing to perform any calculation, beforehand, applying part *iv)* of **Property 21** we know that the four tuples are dependent since more than 3 3-tuples are never independent. To check (3), as before, with the tuples placed in columns, their rank should be 3. Now,

$$\text{rango}\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ 3 & 3 & 1 & 5 \end{pmatrix} = 2 < 3$$

so there are tuples of $\mathbb{R}^3$ that cannot be obtained by combining those in this example.

3.3 Rank and Inverse Matrix. Calculation of the inverse matrix via elementary operations

In this section we will study the relationships between the concepts of rank and inverse matrix in the case of square matrices.

Property 66. *Let $A \in \mathcal{M}_n$ be a square matrix of order n. Then,*

$$\exists A^{-1} \Leftrightarrow \text{rango}(A) = n.$$

Let's make some comments about this last property:

- From the previous property it follows that if a matrix has an inverse, its column tuples must be independent. In other words,

$$\exists A^{-1} \Leftrightarrow \text{the columns of } A \text{ are independent.}$$

  This equivalent statement can also be written in terms of rows.

- As a consequence of **Corollary 17**, a square matrix $A \in \mathcal{M}_n$ of order n (with n rows and n columns) will have rank at most n. Then, n is the maximum value that the rank of A can take and, consequently, it is said that A has maximum rank when $\text{rango}(A) = n$. Using this nomenclature, **Property 23** can be rewritten as

$$\exists A^{-1} \Leftrightarrow A \text{ has maximum rank.}$$

3.3.1 Calculation of the inverse matrix via elementary operations

Let's take a square matrix of order n, $A \in \mathcal{M}_n$. If the matrix has rank n then

$$A \xrightarrow[\text{elementary operations}]{} I_n.$$

Let's form the matrix $(A|I_n)$ which is obtained by appending the identity matrix to the matrix A. If the elementary operations from before are all by rows

$$(A|I_n) \xrightarrow[\text{elementary operations}]{} (I_n|B).$$

It can be checked that the matrix B that appears in this way stores the elementary operations that we have applied to A in the sense

$$\underbrace{A \xrightarrow[\text{el. op.}]{} I_n}_{\text{the el. op. transform } A \text{ into } I_n} \qquad \text{and therefore} \qquad \underbrace{A \cdot B = I_n}_{\text{the product by } B \text{ transforms } A \text{ into } I_n}.$$

But this last equality indicates to us that the matrix B is precisely the inverse matrix of A and therefore $A^{-1} = B$.

Property 67. *Suppose that the matrix $A \in \mathcal{M}_n$ has an inverse and that L is a list of elementary operations by rows that transforms A into I_n, then if we apply the operations of L to the block matrix $(A|I_n)$ we will obtain*

$$(A|I_n) \xrightarrow[\textit{Operations } L]{} (I_n|B),$$

where B is the inverse of A.

In this way, from **Property 24** it follows that to calculate the inverse of the matrix $A \in \mathcal{M}_n$, we must reduce A to I_n but now applying elementary operations only by rows. For the calculation of the rank we can apply operations both by rows and by columns but now, to obtain the inverse, we have to limit ourselves only to operations by rows. However, this is still possible by applying the Gaussian elimination method that we saw on page 106. We will only have to take into account the following points whose objective is to avoid in all steps the performance of operations by columns:

a) Since we can only apply operations by rows, in step 1 we will only select columns since to cancel the elements of a column the operations to perform are by rows.

b) In step 2, we must take the precaution of selecting an element that is not at the height of the pivots selected in previous steps.

c) We will omit step 4 since it involves performing operations by columns.

d) Once steps 1 to 5 have been applied for all the columns, it will be sufficient to divide the rows in order to transform all the non-zero elements into ones. Finally we will order the rows to obtain the identity.

Examples 68.

1) Calculate the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & 2 & 3 \end{pmatrix}.$$

To do this, we form the matrix $(A \mid I_4)$ and we will apply elementary operations by rows to it until we transform it into $(I_4 \mid B)$.

$$\left(\begin{array}{cccc|cccc} 1 & 2 & \mathbf{0} & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & \mathbf{2} & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & \overline{\underline{\mathbf{1}}} & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & \mathbf{2} & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{\text{F2=F2-2F3} \\ \text{F4=F4-2F3}}]{} \left(\begin{array}{cccc|cccc} 1 & 2 & \mathbf{0} & 1 & 1 & 0 & 0 & 0 \\ -3 & -1 & \mathbf{0} & 1 & 0 & 1 & -2 & 0 \\ 2 & 0 & \overline{\underline{\mathbf{1}}} & 0 & 0 & 0 & 1 & 0 \\ -3 & -1 & \mathbf{0} & 3 & 0 & 0 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & \mathbf{1} & 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & \overline{\underline{\mathbf{1}}} & 0 & 1 & -2 & 0 \\ 2 & 0 & \overline{\underline{1}} & \mathbf{0} & 0 & 0 & 1 & 0 \\ -3 & -1 & 0 & \mathbf{3} & 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow[\substack{\text{F1=F1-F2} \\ \text{F4=F4-3F2}}]{} \left(\begin{array}{cccc|cccc} 4 & 3 & 0 & \mathbf{0} & 1 & -1 & 2 & 0 \\ -3 & -1 & 0 & \overline{\underline{\mathbf{1}}} & 0 & 1 & -2 & 0 \\ 2 & 0 & \overline{\underline{1}} & \mathbf{0} & 0 & 0 & 1 & 0 \\ 6 & 2 & 0 & \mathbf{0} & 0 & -3 & 4 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}\text{F4}]{}$$

$$\left(\begin{array}{cccc|cccc} 4 & \mathbf{3} & 0 & 0 & 1 & -1 & 2 & 0 \\ -3 & -\mathbf{1} & 0 & \overline{\underline{1}} & 0 & 1 & -2 & 0 \\ 2 & \mathbf{0} & \overline{\underline{1}} & 0 & 0 & 0 & 1 & 0 \\ 3 & \overline{\underline{\mathbf{1}}} & 0 & 0 & 0 & \frac{-3}{2} & 2 & \frac{1}{2} \end{array}\right) \xrightarrow[\substack{\text{F1=F1-3F4} \\ \text{F2=F2+F4}}]{} \left(\begin{array}{cccc|cccc} -5 & \mathbf{0} & 0 & 0 & 1 & \frac{7}{2} & -4 & \frac{-3}{2} \\ 0 & \mathbf{0} & 0 & \overline{\underline{1}} & 0 & \frac{-1}{2} & 0 & \frac{1}{2} \\ 2 & \mathbf{0} & \overline{\underline{1}} & 0 & 0 & 0 & 1 & 0 \\ 3 & \overline{\underline{\mathbf{1}}} & 0 & 0 & 0 & \frac{-3}{2} & 2 & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow[\frac{-1}{5}\text{F1}]{}$$

$$\left(\begin{array}{cccc|cccc} \overline{\underline{\mathbf{1}}} & 0 & 0 & 0 & \frac{-1}{5} & \frac{-7}{10} & \frac{4}{5} & \frac{3}{10} \\ \mathbf{0} & 0 & 0 & \overline{\underline{1}} & 0 & \frac{-1}{2} & 0 & \frac{1}{2} \\ \mathbf{2} & 0 & \overline{\underline{1}} & 0 & 0 & 0 & 1 & 0 \\ \mathbf{3} & \overline{\underline{1}} & 0 & 0 & 0 & \frac{-3}{2} & 2 & \frac{1}{2} \end{array}\right) \xrightarrow[\substack{\text{F3=F3-2F1} \\ \text{F4=F4-3F1}}]{} \left(\begin{array}{cccc|cccc} \overline{\underline{\mathbf{1}}} & 0 & 0 & 0 & \frac{-1}{5} & \frac{-7}{10} & \frac{4}{5} & \frac{3}{10} \\ \mathbf{0} & 0 & 0 & \overline{\underline{1}} & 0 & \frac{-1}{2} & 0 & \frac{1}{2} \\ \mathbf{0} & 0 & \overline{\underline{1}} & 0 & \frac{2}{5} & \frac{7}{5} & \frac{-3}{5} & \frac{-3}{5} \\ \mathbf{0} & \overline{\underline{1}} & 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{-2}{5} & \frac{-2}{5} \end{array}\right)$$

$$\xrightarrow[\text{reordering}]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-7}{10} & \frac{4}{5} & \frac{3}{10} \\ 0 & 1 & 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{-2}{5} & \frac{-2}{5} \\ 0 & 0 & 1 & 0 & \frac{2}{5} & \frac{7}{5} & \frac{-3}{5} & \frac{-3}{5} \\ 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} & 0 & \frac{1}{2} \end{array}\right)$$

from which we obtain,

$$A^{-1} = \begin{pmatrix} \frac{-1}{5} & \frac{-7}{10} & \frac{4}{5} & \frac{3}{10} \\ \frac{3}{5} & \frac{3}{5} & \frac{-2}{5} & \frac{-2}{5} \\ \frac{2}{5} & \frac{7}{5} & \frac{-3}{5} & \frac{-3}{5} \\ 0 & \frac{-1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

2) Consider $A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{C3=C3-C1}]{} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{C3=C3+C2}]{} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{2}\text{F1}]{} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow[\text{ordering columns}]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Therefore rango$(A) = 3$ and consequently the matrix A is regular. We can now calculate its inverse by applying elementary operations to the rows of the matrix $\left(A \mid I_3 \right)$

$$\left(A \mid I_3 \right) = \left(\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{2}\text{F1}]{} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{F2=F2-F1}]{} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{F3=F3+F1}]{} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{reordering rows}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right) = \left(I_3 \mid B \right).$$

Therefore:

$$A^{-1} = B = \begin{pmatrix} -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

3) Applying elementary transformations to $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ we achieve:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow[\text{F2=F2-F1}]{} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow[\text{C2=C2-C1}]{} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $\text{rango}(A) = 1 < 2$ and consequently A is a singular matrix that does not have an inverse.

Given $A \in \mathcal{M}_n$ we have seen that

$$\text{rango}(A) = n \Leftrightarrow \exists A^{-1}$$

However, if $\text{rango}(A) < n$ or A is not square, this characterization does not make sense.

Definition 69. Given $A \in \mathcal{M}_{m\times n}$, we call a minor of order r of the matrix A any square submatrix of order r of A.

Example 70. Let $A = \begin{pmatrix} 2 & 1 & 3 & 6 \\ -1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ we have that:

- $\star$ $\begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ is a square submatrix of order 3 of A and therefore it is a minor of order 3 of the matrix A.
- $\star$ $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ is a square submatrix of order 2 of A and for that reason it is a minor of order 2 of A.
- $\star$ $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ is a square submatrix of order 2 of A and as a consequence a minor of order 2.
- $\star$ (2), (0) or (4) are square submatrices of order 1, that is, minors of order 1.

Property 71. *Given* $A \in \mathcal{M}_{m\times n}$*,*

$\operatorname{rango}(A) = r \Leftrightarrow$ *the order of the largest regular minor of* A *is* r*.*

In other words, if a matrix has rank r, necessarily we will be able to find within it a square submatrix, a minor, of size r with inverse and furthermore it is not possible to find larger submatrices with inverse.

Aquí está la traducción al inglés del fragmento sobre determinantes, cumpliendo estrictamente con todas las indicaciones.

4 Determinant of a Matrix

For the definition of the determinant we will follow a constructive method based on the formulas for the expansion of the determinant by a row or column.

Definition 72. Given $A \in \mathcal{M}_n$ we define the determinant of A and denote it by $\det(A)$ or $|A|$, as the number that satisfies:

- If $A = (a)_{1\times 1} \in \mathcal{M}_1$, then $|A| = |(a)| = a$.
- If $A = (a_{ij})_{n\times n} \in \mathcal{M}_n$ with $n > 1$, then $|A|$ is defined in either of the following two ways:

  a) For any $i = 1, \ldots, n$:

  $$\begin{aligned} |A| &= a_{i1} \cdot \Delta_{i1} + a_{i2} \cdot \Delta_{i2} + \cdots + a_{in} \cdot \Delta_{in} \\ &= \underbrace{\begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix}}_{\text{row } i \text{ of } A} \cdot \begin{pmatrix} \Delta_{i1} \\ \Delta_{i2} \\ \vdots \\ \Delta_{in} \end{pmatrix}. \end{aligned}$$

  The above formula is known as the expansion of the determinant of the matrix A along the i-th row.

b) For any $j = 1, \dots, n$:

$$\begin{aligned}|A| &= a_{1j} \cdot \Delta_{1j} + a_{2j} \cdot \Delta_{2j} + \cdots + a_{nj} \cdot \Delta_{nj} \\ &= \begin{pmatrix} \Delta_{1j} & \Delta_{2j} & \dots & \Delta_{nj} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}}_{\text{column } j \text{ of } A} .\end{aligned}$$

This formula is called the expansion of the determinant of the matrix A along the j-th column.

Where Δ_{ij} is called the (i, j) cofactor of the matrix and is defined by the formula

$$\Delta_{ij} = (-1)^{i+j} \cdot |A_{(i,j)}|,$$

where $A_{(i,j)}$ is the submatrix of A obtained by deleting the i-th row and the j-th column.

Remark. Although the symbol we use for the determinant of a matrix, $|\cdot|$, is the same as for the absolute value of a real number, they are totally different concepts and have no relation.

Examples 73.
1) To calculate the determinant of the matrix $(3)_{1\times 1}$ we will refer to the first point of the definition of determinant since this matrix is a 1×1 type matrix. Then we have:

$$|(3)| = 3.$$

$|(-5)| = -5$ (remember that $|(-5)|$ is the determinant of the matrix $(-5)_{1\times 1}$ and not the absolute value of the number -5).

2) *Determinant of the square matrix of order 2.*

Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2$:

*
$$A_{(1,1)} = (a_{22}).$$

*
$$A_{(1,2)} = (a_{21}).$$

*
$$A_{(2,1)} = (a_{12}).$$

*
$$A_{(2,2)} = (a_{11}).$$

From the above we calculate the cofactors of the matrix:

$$\begin{aligned} \Delta_{11} &= (-1)^{1+1} \cdot |A_{(1,1)}| = (-1)^2 \cdot |(a_{22})| = a_{22}, \\ \Delta_{12} &= (-1)^{1+2} \cdot |A_{(1,2)}| = (-1)^3 \cdot |(a_{21})| = -a_{21}, \\ \Delta_{21} &= (-1)^{2+1} \cdot |A_{(2,1)}| = (-1)^3 \cdot |(a_{12})| = -a_{12}, \\ \Delta_{22} &= (-1)^{2+2} \cdot |A_{(2,2)}| = (-1)^4 \cdot |(a_{11})| = a_{11}. \end{aligned}$$

We will calculate following two of the previous possibilities:

$$\begin{aligned}
\left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right| &= \\
\text{(row 1)} &= a_{11}\cdot\Delta_{11} + a_{12}\cdot\Delta_{12} = a_{11}\cdot a_{22} + a_{12}\cdot(-a_{21}) \\
&= a_{11}\cdot a_{22} - a_{12}\cdot a_{21}. \\
\text{(column 2)} &= a_{12}\cdot\Delta_{12} + a_{22}\cdot\Delta_{22} = a_{12}\cdot(-a_{21}) + a_{22}\cdot a_{11} \\
&= a_{11}\cdot a_{22} - a_{12}\cdot a_{21}.
\end{aligned}$$

In summary:

$$\left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right| = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}.$$

Schematically,

$$\left|\begin{pmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{pmatrix}\right| = \begin{pmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{pmatrix},$$

where the elements that appear joined by a segment must be multiplied.

3) *Determinant of a square matrix of order 3.*

Given the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we will calculate its determinant by expanding along the first column. We will then have:

$$|A| = a_{11} \cdot \Delta_{11} + a_{21} \cdot \Delta_{21} + a_{31} \cdot \Delta_{31}.$$

We next calculate the necessary cofactors:

$$\begin{aligned} \Delta_{11} &= (-1)^{1+1}|A_{(1,1)}| = \left|\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\right| = a_{22} \cdot a_{33} - a_{23} \cdot a_{32}, \\ \Delta_{21} &= (-1)^{2+1}|A_{(2,1)}| = -\left|\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}\right| = \text{-}a_{12} \cdot a_{33} + a_{32} \cdot a_{13}, \\ \Delta_{31} &= (-1)^{3+1}|A_{(3,1)}| = \left|\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}\right| = a_{12} \cdot a_{23} - a_{22} \cdot a_{13}. \end{aligned}$$

We then have:

$$\begin{aligned} |A| &= a_{11} \cdot (a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) + a_{21} \cdot (a_{32} \cdot a_{13} - a_{12} \cdot a_{33}) \\ &\quad + a_{31} \cdot (a_{12} \cdot a_{23} - a_{22} \cdot a_{13}). \\ &= a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} \\ &\quad -(a_{31} \cdot a_{22} \cdot a_{13} + a_{11} \cdot a_{23} \cdot a_{32} + a_{21} \cdot a_{12} \cdot a_{33}). \end{aligned}$$

The above formula for the determinant of a 3×3 matrix is called Sarrus' rule and can be schematized in the following diagram:

$$\begin{aligned} |A| &= a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} \\ &\quad -(a_{31} \cdot a_{22} \cdot a_{13} + a_{11} \cdot a_{23} \cdot a_{32} + a_{21} \cdot a_{12} \cdot a_{33}) \end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

4) Next we calculate some determinants of 2×2 matrices using the formula obtained in example 3:

- $\star \left| \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \right| = 1 \cdot 0 - 2 \cdot (-1) = 2.$
- $\star \left| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right| = 1 \cdot 1 - 1 \cdot 1 = 0.$
- $\star \left| \begin{pmatrix} 1 & 6 \\ 3 & 9 \end{pmatrix} \right| = 1 \cdot 9 - 6 \cdot 3 = -9.$

5) We now use Sarrus' formula:

$$
\begin{aligned}
\star \left| \begin{pmatrix} 1 & 6 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right| &= \\
&= 1 \cdot 2 \cdot 1 + (-1) \cdot 0 \cdot 0 + 6 \cdot 0 \cdot 1 \\
&\quad -(0 \cdot 2 \cdot 1 + 0 \cdot 0 \cdot 1 + (-1) \cdot 6 \cdot 1) \\
&= 2 + 6 = 8. \\
\star \left| \begin{pmatrix} 4 & 2 & 6 \\ 0 & 3 & -1 \\ 0 & 0 & 5 \end{pmatrix} \right| &= \\
&= 4 \cdot 3 \cdot 5 + 0 \cdot 0 \cdot 6 + 2 \cdot (-1) \cdot 0 \\
&\quad -(0 \cdot 3 \cdot 6 + 0 \cdot (-1) \cdot 4 + 0 \cdot 2 \cdot 5) \\
&= 4 \cdot 3 \cdot 5 = 60.
\end{aligned}
$$

6) $$\left|\begin{pmatrix}1 & 2 & 0 & 1\\ 0 & -1 & 1 & 0\\ 0 & 3 & 0 & 2\\ 1 & 1 & -1 & 0\end{pmatrix}\right| = 0\cdot\Delta_{13} + 1\cdot\Delta_{23} + 0\cdot\Delta_{33} - \Delta_{43}$$

$$= (-1)^{2+3}\left|\begin{pmatrix}1 & 2 & 1\\ 0 & 3 & 2\\ 1 & 1 & 0\end{pmatrix}\right| - (-1)^{4+3}\left|\begin{pmatrix}1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 3 & 2\end{pmatrix}\right|$$

$$= -(0+0+4-0-3-2) + (-2+0+0-0-0-0)$$

$$= 1 - 2 = -1.$$

7) In the following example we solve a determinant of type 4×4 by an expansion along the second row:

$$
\begin{aligned}
\left|\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix}\right| = & \\
&= 2 \cdot \Delta_{21} + 0 \cdot \Delta_{22} + \Delta_{23} + \Delta_{24} \\
&= -2 \cdot \left|\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right| - \left|\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\right| + \left|\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\right| \\
&= -2 \cdot (4 + 2 - 1 - 1 - 4 + 2) - (2 + 1 - 2 + 1 - 1 - 4) \\
&\quad +(2 - 1 - 2 - 1 - 1 - 4) = -4 + 3 - 7 = -8.
\end{aligned}
$$

In the previous examples it becomes clear that the determinant can take any real number as its value. On the other hand, the rank will always be a natural number.

Propiedades 2.

1. *Given* $A = (a_{ij})_{n\times n} \in \mathcal{M}_n$ *a diagonal, lower triangular or upper triangular matrix, it holds that:*

$$|A| = a_{11} \cdot a_{22} \cdot a_{33} \cdot \cdots \cdot a_{nn}.$$

   In particular $|I_n| = 1$.

2. *Given* $A \in \mathcal{M}_n$, $|A^t| = |A|$.

3. *Given* $A, B \in \mathcal{M}_n$, $|A \cdot B| = |A| \cdot |B|$.

4. *Given* $A \in \mathcal{M}_n$ *regular, we have that* $|A| \neq 0$ *and also*

$$|A^{-1}| = \frac{1}{|A|}.$$

5. *Given* $A \in \mathcal{M}_n$ *such that* $|A| \neq 0$ *it holds that* A *is regular.*

6. *Given a matrix* $B \in \mathcal{M}_{m\times n}$,

$$\mathrm{rango}(B) = r \Leftrightarrow \textit{The order of the largest minor of } A \textit{ with non-zero determinant is } r.$$

Examples 74.

1) Considering the matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ we have:

$$|A| = 1 \cdot 3 - 2 \cdot (-1) = 5 \neq 0.$$

Since the determinant of A is non-zero we know that it is a regular matrix and also without needing to know its inverse, A^{-1}, we know the value of its determinant:

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{5}.$$

2) $\left| \begin{pmatrix} 2 & 1 & 3 \\ 0 & -3 & 4 \\ 0 & 0 & 6 \end{pmatrix} \right| = \begin{pmatrix} \text{using} \\ \text{property 1} \end{pmatrix} = 2 \cdot (-3) \cdot 6 = -36.$

3) Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{pmatrix}$ and let's calculate its rank using the two methods we know:

i) Let's see if A has any non-zero minor of order 3:

$$\left\{ \begin{array}{ll} \left| \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 5 \end{pmatrix} \right| = 0 & \left| \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 4 & 2 \end{pmatrix} \right| = 0 \\ & \\ \left| \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 5 & 2 \end{pmatrix} \right| = 0 & \left| \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ 4 & 5 & 2 \end{pmatrix} \right| = 0 \end{array} \right. .$$

However,

$$\left| \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right| = 1 \neq 0.$$

In this way the order of the largest minor with non-zero determinant is 2 and therefore the rank of the matrix will be:

$$\text{rango}(A) = 2.$$

ii) Performing operations on A we have:

$$A \xrightarrow[\text{C3=C3+C1}]{} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 6 & 2 \end{pmatrix} \xrightarrow[\text{C2=C2-2C1}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 6 & 2 \end{pmatrix}$$

$$\xrightarrow[\text{F3=F3-F1}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \end{pmatrix} \xrightarrow[\text{C3=C3-3C2}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{F3=F3-2F2}]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$\text{rango}(A) = \text{rango} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2.$$

Propiedades 3 (Elementary operations for determinants)**.** *Given $A = (a_{ij})_{n\times n} \in \mathcal{M}_n$ we have that:*

1. *A number that multiplies an entire row or an entire column can be taken out of the determinant. That is, $\forall i, j \in \{1, \dots, n\}$*

$$\left|\begin{pmatrix} a_{11} & \cdots & r\cdot a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & r\cdot a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & r\cdot a_{nj} & \cdots & a_{nn} \end{pmatrix}\right| = r\cdot\left|\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}\right|$$

$$\left|\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ r\cdot a_{i1} & r\cdot a_{i2} & \cdots & r\cdot a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right| = r\cdot\left|\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right|.$$

2. *If we interchange one column (respectively row) with another column (resp. row) that is adjacent, the determinant changes sign.*

3. *If to one column (respectively row) we add another column (resp. row) multiplied by a number, the determinant does not change.*

Examples 75.

1. Applying property 1 we have:

   ⋆ $$\left|\begin{pmatrix}1 & 2 & 0\\ 6 & 4 & 3\\ 2 & 8 & 1\end{pmatrix}\right| = \left|\begin{pmatrix}1 & 2\cdot 1 & 0\\ 6 & 2\cdot 2 & 3\\ 2 & 2\cdot 4 & 1\end{pmatrix}\right| = 2\cdot\left|\begin{pmatrix}1 & 1 & 0\\ 6 & 2 & 3\\ 2 & 4 & 1\end{pmatrix}\right|.$$

   ⋆ $$\left|\begin{pmatrix}1 & 2 & 3\\ 0 & \frac{2}{5} & \frac{4}{5}\\ 2 & -9 & 1\end{pmatrix}\right| = \left|\begin{pmatrix}1 & 2 & 3\\ \frac{1}{5}\cdot 0 & \frac{1}{5}\cdot 2 & \frac{1}{5}\cdot 4\\ 2 & -9 & 1\end{pmatrix}\right| = \frac{1}{5}\cdot\left|\begin{pmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 2 & -9 & 1\end{pmatrix}\right|.$$

2. Applying property 2 repeatedly we have:

$$
\begin{aligned}
\left|\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\right| &= (\mathrm{C2}\leftrightarrow\mathrm{C3}) = -\left|\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}\right| \\
&= (\mathrm{C3}\leftrightarrow\mathrm{C4}) = \left|\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\right| \\
&= (\mathrm{C2}\leftrightarrow\mathrm{C3}) = -\left|\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right| = -|I_n| = -1.
\end{aligned}
$$

The last property together with the others we have seen for determinants allow us to solve some simple determinants directly. Let's see it in the following note:

Remark. It is important to keep in mind the following points:

- If $A \in \mathcal{M}_n$ has an entire column of zeros or an entire row of zeros then $|A| = 0$.
- If $A \in \mathcal{M}_n$ has two equal columns or two equal rows then $|A| = 0$.
- If some column (respectively row) can be obtained by adding the other columns (resp. rows) multiplied by numbers, then $|A| = 0$.

Example 76. In the matrix of the following determinant the third column can be obtained as the sum of twice the first column plus three times the second, so:

$$\left|\begin{pmatrix} 1 & 1 & 5 \\ -1 & 6 & 16 \\ 0 & 4 & 12 \end{pmatrix}\right| = (\text{C3=C3-2C1}) = \left|\begin{pmatrix} 1 & 1 & 3 \\ -1 & 6 & 18 \\ 0 & 4 & 12 \end{pmatrix}\right|$$

$$= (\text{C3=C3-3C2}) = \left|\begin{pmatrix} 1 & 1 & 0 \\ -1 & 6 & 0 \\ 0 & 4 & 0 \end{pmatrix}\right| = 0.$$

In practice, for the simplification of a determinant one can, with the necessary precautions, use the procedure for reduction of matrices seen in Section 3.2 with the exception that in the case of determinants we must take into account the following points:

- The multiplication of a row or column by a non-zero number will cause the value of the determinant to change but in its place one can apply part 1 of **Properties 33**.
- The modification of the order of the rows or columns implies a change of sign of the determinant as indicated in part 2 of **Properties 33**.
- Once a row or column has been canceled using a pivot, we will expand the determinant along that row or column to obtain a determinant of smaller size.

Let's see some examples of this method next:

Examples 77.
1)

$$
\begin{aligned}
&\left|\begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & -1 & 0 & 2 \\ 1 & 1 & 4 & 6 \\ 0 & -1 & 1 & 0 \end{pmatrix}\right| = \\
&= \text{(C2=C2+C3)} = \left|\begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & -1 & 0 & 2 \\ 1 & 5 & 4 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right| \\
&= \begin{pmatrix}\text{expanding along} \\ \text{row 4}\end{pmatrix} = (-1)^{4+3} \cdot \left|\begin{pmatrix} 1 & 1 & 3 \\ 3 & -1 & 2 \\ 1 & 5 & 6 \end{pmatrix}\right| \\
&= -\big((-6+45+2) - (-3+10+18)\big) = -16.
\end{aligned}
$$

2)

$$\left|\begin{pmatrix}-1 & 2 & -1 & 2 & 1\\ 2 & 3 & 1 & 4 & 1\\ 2 & 1 & 1 & 4 & 3\\ 6 & 2 & 1 & -1 & 9\\ 1 & -2 & 3 & 6 & 4\end{pmatrix}\right| =$$

$$= \begin{pmatrix}\text{F2=F2+F1}\\ \text{F3=F3+F1}\\ \text{F4=F4+F1}\\ \text{F5=F5+3F1}\end{pmatrix} = \left|\begin{pmatrix}-1 & 2 & -1 & 2 & 1\\ 1 & 5 & 0 & 6 & 2\\ 1 & 3 & 0 & 6 & 4\\ 5 & 4 & 0 & 1 & 10\\ -2 & 4 & 0 & 12 & 7\end{pmatrix}\right|$$

$$= \begin{pmatrix}\text{expanding along}\\ \text{column 3}\end{pmatrix} = (-1)\cdot(-1)^{1+3}\cdot\left|\begin{pmatrix}1 & 5 & 6 & 2\\ 1 & 3 & 6 & 4\\ 5 & 4 & 1 & 10\\ -2 & 4 & 12 & 7\end{pmatrix}\right|$$

$$= \text{(F2=F2-F1)} = -\left|\begin{pmatrix}1 & 5 & 6 & 2\\ 0 & -2 & 0 & 2\\ 5 & 4 & 1 & 10\\ -2 & 4 & 12 & 7\end{pmatrix}\right|$$

$$= \text{(C2=C2+C4)} = -\left|\begin{pmatrix}1 & 7 & 6 & 2\\ 0 & 0 & 0 & 2\\ 5 & 14 & 1 & 10\\ -2 & 11 & 12 & 7\end{pmatrix}\right|$$

$$
\begin{aligned}
&= \left(\begin{matrix}\text{expanding along}\\ \text{row } 2\end{matrix}\right) = -2 \cdot (-1)^{2+4} \cdot \left| \begin{pmatrix} 1 & 7 & 6 \\ 5 & 14 & 1 \\ -2 & 11 & 12 \end{pmatrix} \right| \\
&= -2 \cdot \big((168 + 330 - 14) - (-168 + 11 + 420)\big) \\
&= -2 \cdot 221 = -442.
\end{aligned}
$$

4.1 Calculation of the inverse via determinants

In this section we will study an alternative method for the calculation of the inverse of a matrix based on the concept of determinant.

Definition 78. Given $A \in \mathcal{M}_n$, we call the adjugate matrix of A and denote it by $Adj(A)$, the matrix:

$$Adj(A) = (\Delta_{ij})_{n\times n} \in \mathcal{M}_n.$$

The adjugate matrix of A is, therefore, the matrix formed by all the cofactors of A arranged in order.

Property 79 (Calculation of the inverse via determinants)**.** *Let $A \in \mathcal{M}_n$ such that $\det(A) \neq 0$, then A is a regular matrix and also:*

$$A^{-1} = \frac{1}{|A|} \cdot (\mathrm{Adj}(A))^t.$$

Examples 80.

1) Let $A = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}$, then

$$|A| = 5 + 8 = 13 \neq 0,$$

so A is a regular matrix and we can calculate its inverse for which we will first obtain the cofactors of the matrix

$$\begin{array}{ll} \Delta_{11} = (-1)^{1+1}|(5)| = 5, & \Delta_{12} = (-1)^{1+2}|(4)| = -4, \\ \Delta_{21} = (-1)^{2+1}|(-2)| = 2, & \Delta_{22} = (-1)^{2+2}|(1)| = 1 \end{array}$$

and then the adjugate matrix will be:

$$\mathrm{Adj}(A) = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}.$$

Finally the inverse is:

$$A^{-1} = \frac{1}{|A|} \cdot \mathrm{Adj}(A)^t = \frac{1}{13} \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}^t = \frac{1}{13} \begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix}.$$

2) Let $A = \begin{pmatrix} 1 & 6 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ then its determinant is

$$|A| = -2 - 1 - 6 = -9$$

and also its cofactors are

$$\begin{cases} \Delta_{11} = \left|\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\right| = -3, & \Delta_{12} = -\left|\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right| = -1, & \Delta_{13} = \left|\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\right| = -1, \\ \Delta_{21} = -\left|\begin{pmatrix} 6 & 0 \\ 1 & -1 \end{pmatrix}\right| = 6, & \Delta_{22} = \left|\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right| = -1, & \Delta_{23} = -\left|\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\right| = -1, \\ \Delta_{31} = \left|\begin{pmatrix} 6 & 0 \\ 2 & 1 \end{pmatrix}\right| = 6, & \Delta_{32} = -\left|\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right| = -1, & \Delta_{33} = \left|\begin{pmatrix} 1 & 6 \\ -1 & 2 \end{pmatrix}\right| = 8 \end{cases}$$

so its adjugate matrix will be:

$$\mathrm{Adj}(A) = \begin{pmatrix} -3 & -1 & -1 \\ 6 & -1 & -1 \\ 6 & -1 & 8 \end{pmatrix}$$

and A^{-1} will be calculated as:

$$A^{-1} = \frac{1}{|A|} \cdot \mathrm{Adj}(A)^t = \frac{1}{-9} \cdot \begin{pmatrix} -3 & 6 & 6 \\ -1 & -1 & -1 \\ -1 & -1 & 8 \end{pmatrix}.$$

Aquí está la traducción al inglés del fragmento final del capítulo de matrices, cumpliendo estrictamente con todas las indicaciones.

“‘latex

5 Additional Material

5.1 Rows and columns of proportions

Extension of concepts about product of a number by a matrix. Page 34

We have seen that when several data a_1, a_2,..., a_n intervene in a certain phenomenon, we can represent this information by an element of $\mathbb{R}^n$ in the form,

$$(a_1, a_2, \ldots, a_n).$$

It is frequent that it is of interest to determine what percentage each quantity represents with respect to the total. This can be done using percentages or fractions (rates per one):

- The calculation of percentages for each quantity of $(a_1, a_2, \ldots, a_n)$ is performed in the following way:
  - Percentage of a_1 with respect to the total= $\dfrac{100}{a_1 + a_2 + \cdots + a_n} a$
  - Percentage of a_2 with respect to the total= $\dfrac{100}{a_1 + a_2 + \cdots + a_n} a$

- $\vdots$
- Percentage of a_n with respect to the total$= \dfrac{100}{a_1 + a_2 + \cdots + a_n} a$

- The calculation of fractions (rates per one) for each quantity of $(a_1, a_2, \ldots, a_n)$ is performed as follows:
  - Fraction (rate per one) of a_1 with respect to the total= $\dfrac{1}{a_1 + a_2 + \cdots + a_n} a_1$.
  - Fraction (rate per one) of a_2 with respect to the total= $\dfrac{1}{a_1 + a_2 + \cdots + a_n} a_2$.
  - $\vdots$
  - Fraction (rate per one) of a_n with respect to the total= $\dfrac{1}{a_1 + a_2 + \cdots + a_n} a_n$.

Then,

$$(\frac{100}{a_1 + a_2 + \cdots + a_n} a_1, \ldots, \frac{100}{a_1 + a_2 + \cdots + a_n} a_n) =$$

$$= \frac{100}{a_1 + a_2 + \cdots + a_n} (a_1, a_2, \ldots, a_n).$$

And also

$$(\frac{1}{a_1 + a_2 + \cdots + a_n} a_1, \ldots, \frac{1}{a_1 + a_2 + \cdots + a_n} a_n) =$$

$$= \frac{1}{a_1 + a_2 + \cdots + a_n} (a_1, a_2, \ldots, a_n).$$

In other words,

The tuple of corresponding percentages is:	$\dfrac{100}{a_1 + a_2 + \cdots + a_n}(a_1, a_2, \ldots, a_n)$
The tuple of corresponding fractions (rates per one) is:	$\dfrac{1}{a_1 + a_2 + \cdots + a_n}(a_1, a_2, \ldots, a_n)$

Ejemplo 4. The consumption of raw materials in a certain industrial zone is given by: 1123 tons of steel, 820 tons of aluminum, 530 tons of plastic materials. We can represent this distribution by the 3-tuple

$$(1123, 820, 530) \in \mathbb{R}^3.$$

Let's calculate the percentages:

- $\%$ of steel $= 100\dfrac{1123}{1123+820+530} = 45.41\%.$
- $\%$ of aluminum $= 100\dfrac{820}{1123+820+530} = 33.15\%.$
- $\%$ of materials $= 100\dfrac{530}{1123+820+530} = 21.43\%.$

As we have just seen,

$$\frac{100}{1123+820+530}(1123, 820, 530) = (45.41, 33.15, 21.43).$$

In the same way the tuple of fractions (rates per one) will be,

$$\frac{1}{1123+820+530}(1123, 820, 530) = (0.4541, 0.3315, 0.2143).$$

6 Simplification rules and matrix equalities

Extension on operations and simplifications with matrices. Page 56

Propiedades 5.

1. *Let* $A, B, C \in \mathcal{M}_{m\times n}$, *then:*
$$A + B = A + C \;\Rightarrow\; B = C,$$
$$A + B = C \;\Rightarrow\; B = C - A.$$
2. *Let* $A \in \mathcal{M}_n$ *be a regular matrix and* $B, C \in \mathcal{M}_{m\times n}$, *then:*
$$B \cdot A = C \cdot A \;\Rightarrow\; B = C,$$
$$B \cdot A = C \;\Rightarrow\; B = C \cdot A^{-1}.$$
3. *Let* $A \in \mathcal{M}_m$ *be a regular matrix and* $B, C \in \mathcal{M}_{m\times n}$, *then:*
$$A \cdot B = A \cdot C \;\Rightarrow\; B = C,$$
$$A \cdot B = C \;\Rightarrow\; B = A^{-1} \cdot C.$$
4. *Let* $r \in \mathbb{R}$, $r \neq 0$ *and* $B, C \in \mathcal{M}_{m\times n}$, *then:*
$$r \cdot B = r \cdot C \;\Rightarrow\; B = C,$$
$$r \cdot B = C \;\Rightarrow\; B = \frac{1}{r} \cdot C.$$

Remark.

- $$\underbrace{\begin{pmatrix}1&1\\1&1\end{pmatrix}\cdot\begin{pmatrix}1&2\\-1&-2\end{pmatrix}}_{=0_{2\times 2}}=\underbrace{\begin{pmatrix}1&1\\1&1\end{pmatrix}\cdot\begin{pmatrix}-3&1\\3&-1\end{pmatrix}}_{=0_{2\times 2}}$$

however,

$$\cancel{\begin{pmatrix}1&1\\1&1\end{pmatrix}}\cdot\begin{pmatrix}1&2\\-1&-2\end{pmatrix}=\cancel{\begin{pmatrix}1&1\\1&1\end{pmatrix}}\cdot\begin{pmatrix}-3&1\\3&-1\end{pmatrix}\Rightarrow\begin{pmatrix}1&2\\-1&-2\end{pmatrix}=\begin{pmatrix}-3&1\\3&-1\end{pmatrix}$$

- Given $A, B \in \mathcal{M}_n$, we cannot perform the simplification

$$\cancel{A}\cdot B\cdot\cancel{A}^{-1}=B.$$

7 Matrix models based on powers of matrices

Extension of concepts about powers of matrices. Page 56

The product and the power of matrices are fundamental in the formulation of the most important matrix models.

Suppose we are studying a phenomenon in which several magnitudes $a_1, a_2, \ldots, a_k$ intervene that vary with respect to time. If we arrange in tuple form the value of the magnitudes in each period n, we will obtain a list of k-tuples, P_0, P_1, $\ldots$, P_n that provide us with the information of the phenomenon in each period.

In numerous situations we can calculate those tuples by means of a formula of the type

$$P_n = A^n \cdot P_0,$$

where A is a square matrix of order k which is called the **transition matrix**.

We will illustrate this better next with a classic example of a matrix model based on the powering of matrices.

Ejemplo 6. Suppose that in a certain commercial sector three companies compete, which we will call A, B and C. From one year to the next,

	Customers of A	**Customers of B**	**Customers of C**
Switch to A	80%	10%	10%
Switch to B	10%	60%	20%
Switch to C	10%	30%	70%

Suppose also that in the year the studies began, the company A had 210 customers, B had 190 and C, 320.

Assuming that the year $k = 0$ is the year the study of the customers of the three companies began, we will call:

- A_k = customers in company A after k years.
- B_k = customers in company B after k years.
- C_k = customers in company C after k years.

The information for each year will be grouped into a column tuple:

$$P_k = \begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix}.$$

According to the problem data

$$P_0 = \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix}.$$

Applying the transition table,

- $$\underbrace{A_{k+1}}_{\text{customers in A in year } k+1} = 80\% \text{ of } A_k + 10\% \text{ of } B_k + 10\% \text{ of } C_k$$
  $$= 0.8A_k + 0.1B_k + 0.1C_k.$$

- $$\underbrace{B_{k+1}}_{\text{customers in B in year } k+1} = 10\% \text{ of } A_k + 60\% \text{ of } B_k + 20\% \text{ of } C_k$$
  $$= 0.1A_k + 0.6B_k + 0.2C_k.$$

- $$\underbrace{C_{k+1}}_{\text{customers in C in year } k+1} = 10\% \text{ of } A_k + 30\% \text{ of } B_k + 70\% \text{ of } C_k$$
  $$= 0.1A_k + 0.3B_k + 0.7C_k$$

Wsing the definition of the product of matrices, it is easy to realize that

$$P_{k+1} = \begin{pmatrix} 0.8A_k + 0.1B_k + 0.1C_k \\ 0.1A_k + 0.6B_k + 0.2C_k \\ 0.1A_k + 0.3B_k + 0.7C_k \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \cdot P_k.$$

Calling $A = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix}$,

$$P_{k+1} = AP_k.$$

We therefore have,

$$\begin{array}{rcl} P_1 & = & AP_0 \\ P_2 & = & AP_1 \\ P_3 & = & AP_2 \\ P_4 & = & AP_3 \\ & \text{etc.} & \end{array}$$

Then, if we want to calculate P_4

Then, if we want to calculate P_4

$$P_1 = AP_0$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= AP_1 \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= A(AP_0) \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 \end{aligned}$$

Then, if we want to calculate P_4

$$
\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2 P_0
\end{aligned}
$$

Then, if we want to calculate P_4

$$\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2P_0 \\
P_3 &= AP_2
\end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 = A^2P_0 \\ P_3 &= A(A^2P_0) \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 = A^2 P_0 \\ P_3 &= (AA^2)P_0 \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2P_0 \\
P_3 &= (AA^2)P_0 = A^3P_0
\end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 = A^2 P_0 \\ P_3 &= (AA^2)P_0 = A^3 P_0 \\ P_4 &= AP_3 \end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2 P_0 \\
P_3 &= (AA^2)P_0 = A^3 P_0 \\
P_4 &= A(A^3 P_0)
\end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2P_0 \\
P_3 &= (AA^2)P_0 = A^3P_0 \\
P_4 &= (AA^3)P_0
\end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2 P_0 \\
P_3 &= (AA^2)P_0 = A^3 P_0 \\
P_4 &= (AA^3)P_0 = A^4 P_0
\end{aligned}$$

Then, if we want to calculate P_4

$$\begin{aligned} P_1 &= AP_0 \\ P_2 &= (AA)P_0 = A^2P_0 \\ P_3 &= (AA^2)P_0 = A^3P_0 \\ P_4 &= (AA^3)P_0 = A^4P_0 \end{aligned}$$

Then, in general,

$$\boxed{P_k = A^kP_0}\ .$$

Then, if we want to calculate P_4

$$\begin{aligned}
P_1 &= AP_0 \\
P_2 &= (AA)P_0 = A^2P_0 \\
P_3 &= (AA^2)P_0 = A^3P_0 \\
P_4 &= (AA^3)P_0 = A^4P_0
\end{aligned}$$

Then, in general,

$$\boxed{P_k = A^kP_0}\ .$$

The matrix A regulates the passage from one year to the next and is the **transition matrix** for this problem.

Since we know the initial distribution of customers, P_0, we can easily calculate the distribution in successive years. To do this, we calculate several powers of A:

$$\begin{aligned}
A^2 = AA &= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \\
&= \begin{pmatrix} 0.66 & 0.17 & 0.17 \\ 0.16 & 0.43 & 0.27 \\ 0.18 & 0.4 & 0.56 \end{pmatrix}. \\
A^3 = AA^2 &= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.66 & 0.17 & 0.17 \\ 0.16 & 0.43 & 0.27 \\ 0.18 & 0.4 & 0.56 \end{pmatrix} \\
&= \begin{pmatrix} 0.562 & 0.219 & 0.219 \\ 0.198 & 0.355 & 0.291 \\ 0.24 & 0.426 & 0.49 \end{pmatrix}. \\
A^4 = AA^3 &= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.562 & 0.219 & 0.219 \\ 0.198 & 0.355 & 0.291 \\ 0.24 & 0.426 & 0.49 \end{pmatrix} \\
&= \begin{pmatrix} 0.4934 & 0.2533 & 0.2533 \\ 0.223 & 0.3201 & 0.2945 \\ 0.2836 & 0.4266 & 0.4522 \end{pmatrix}.
\end{aligned}$$

Using these calculations with equation (42) we have that

$$P_1 = AP_0 \ = \ \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 219 \\ 199 \\ 302 \end{pmatrix}.$$

$$P_2 = A^2 P_0 \ = \ \begin{pmatrix} 0.66 & 0.17 & 0.17 \\ 0.16 & 0.43 & 0.27 \\ 0.18 & 0.4 & 0.56 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 225.3 \\ 201.7 \\ 293 \end{pmatrix}.$$

$$P_3 = A^3 P_0 \ = \ \begin{pmatrix} 0.562 & 0.219 & 0.219 \\ 0.198 & 0.355 & 0.291 \\ 0.24 & 0.426 & 0.49 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 229.71 \\ 202.15 \\ 288.14 \end{pmatrix}.$$

$$P_4 = A^4 P_0 \ = \ \begin{pmatrix} 0.4934 & 0.2533 & 0.2533 \\ 0.223 & 0.3201 & 0.2945 \\ 0.2836 & 0.4266 & 0.4522 \end{pmatrix} \begin{pmatrix} 210 \\ 190 \\ 320 \end{pmatrix} = \begin{pmatrix} 232.797 \\ 201.889 \\ 285.314 \end{pmatrix}.$$

On the other hand, once this model is set up, several issues arise to be solved:

a) Is it possible to study the future trend in the distribution of customers?

b) Do equilibrium distributions exist?