Chapter 5

Linear Systems

5.1 Basic concepts

An equation is a mathematical equality involving one or more unknown data called unknowns. A system of equations is a set of equations that generally share the same unknowns. In this chapter we will study a particular type of systems, 'linear systems', which are of special importance due to:

- Their wide range of applications. At the base of practically any mathematical calculation there is always the need to solve a linear system.
- The possibility of a theoretical treatment through elementary algebraic techniques. In fact, all the techniques we will use have already been presented in the previous chapter.

Let's start with the fundamental definitions of the theory of linear systems of equations.

Definición 1. A linear system with m equations and n ordered variables or unknowns, $(x_1, x_2, \ldots, x_n)$, is a set of equations of the form

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. ,$$

where for each $i = 1, \ldots, m$, $j = 1, \ldots, n$:

- $a_{ij} \in \mathbb{R}$ is called the (i, j) coefficient of the system.
- $b_i \in \mathbb{R}$ is called the i-th constant term of the system.
- $x_1, x_2, \ldots, x_n$ are called the unknowns of the system and are symbols representing an unknown value to be calculated.

We call a solution of the system any n-tuple of real numbers $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ such that if in the system, for all $i = 1, \ldots, n$, we substitute x_i by s_i, all its equations are true. Solving a system means finding the set of all its solutions.

We will say that the system is:

- **homogeneous** if $\forall i = 1, \ldots, m$, $b_i = 0$.
- **non-homogeneous** if for some $i \in \{1, \ldots, m\}$, $b_i \neq 0$.

In a system, unknown data different from the variables may appear, which we will call the parameters of the system.

Every linear system can be represented as a matrix equation. This will allow us to approach its treatment through the matrix techniques of Chapter 4. In the following definition we establish the corresponding nomenclature.

Definición 2. Given the linear system

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. ,$$

we call:

- Coefficient matrix of the system to the $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m\times n}.$$

- Column of constant terms of the system to $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathcal{M}_{m\times 1}$.

- Column of variables of the system to $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

- Matrix equation of the system or matrix form of the system to

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \Leftrightarrow A \cdot X = B.$$

- Augmented matrix of the system to the matrix

$$A^* = (A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

In the following examples we illustrate the concepts introduced in the previous definitions. We see in them that there are systems that have a unique solution but we can also find others that have multiple solutions. In that latter case, when we have a system with several solutions, it is essential to find an adequate method that allows us to write them all. The basic technique for representing multiple solutions consists of using parameters. In the last examples we present the basic ideas of this technique.

Ejemplos 3.
1) Consider the following system of two equations and variables (x, y, z):

$$\left\{\begin{array}{l} x+2y+z=1 \\ x+y-2z=3 \end{array}\right. .$$

This is a non-homogeneous linear system for which the coefficient matrix and the columns of constant terms and variables are:

$$A=\begin{pmatrix}1 & 2 & 1\\ 1 & 1 & -2\end{pmatrix}, \quad B=\begin{pmatrix}1\\3\end{pmatrix}, \quad X=\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

The matrix equation of the system is

$$\begin{pmatrix}1 & 2 & 1\\ 1 & 1 & -2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

and the augmented matrix is

$$\left(\begin{array}{ccc|c}1 & 2 & 1 & 1\\ 1 & 1 & -2 & 3\end{array}\right).$$

Furthermore, it is easy to check that $(5,-2,0)$, $(10,-5,1)$, $(0,1,-1)$ are solutions of the system. To do this, we substitute the variables x, y, and z by the corresponding values in the system's equations or in its matrix equation. Thus for example:

$$(5,-2,0) \text{ is a solution since taking } \left\{\begin{array}{l} x=5 \\ y=-2 \\ z=0 \end{array}\right. \text{ we have } \left\{\begin{array}{c} \left\{\begin{array}{l} 5+2(-2)+0=1 \\ 5+(-2)-2\ 0=3 \end{array}\right. \\ \text{or} \\ \begin{pmatrix}1 & 2 & 1\\ 1 & 1 & -2\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\0\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix} \end{array}\right. .$$

2) Consider the system of three equations:

$$\left\{\begin{array}{l} \alpha x+3y+z=1 \\ x+y+z=-1 \\ x-y+z=0 \end{array}\right. .$$

Since we have not indicated a list of variables, we must assume that any unknown or unknown data appearing in the system is a variable of it. In the system we find the unknowns

$$\alpha,\ x,\ y,\ z,$$

which, therefore, will all be variables. In this case, the indicated system of equations is not linear because two of its variables, α and x, are multiplied together.

3) Consider the following system of three equations and with variables (x, y, z):

$$\left\{\begin{array}{l} \alpha x+3y+z=1 \\ x+y+z=-1 \\ x-y+z=0 \end{array}\right. .$$

For this system, since we have indicated who its variables are, we know that α is not one of them so it will be a parameter of the system (remember that a parameter is an unknown that appears in the system and is not part of the variable list).

In this case the system is linear and its coefficient matrix and columns of constant terms and variables are:

$$A = \begin{pmatrix} \alpha & 3 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The solution of the system will depend on the value of the parameter α. If $\alpha \neq 1$, it is easy to check that the system has a unique solution given by

$$\left(\frac{3}{\alpha-1}, -\frac{1}{2}, -\frac{1}{2} - \frac{3}{\alpha-1}\right) \text{ or equivalently, } \begin{cases} x = \frac{3}{\alpha-1}, \\ y = -\frac{1}{2}, \\ z = -\frac{1}{2} - \frac{3}{\alpha-1}. \end{cases}$$

On the contrary, if $\alpha = 1$, the system is

$$\begin{cases} x + 3y + z = 1 \\ x + y + z = -1 \\ x - y + z = 0 \end{cases}.$$

and has no solution. We can see this easily because adding the first equation with the third and subtracting twice the second we have

$$\begin{array}{rlcl} & x + 3y + z & = & 1 \\ -2(& x + y + z \quad) & = & -2 \cdot (-1) \\ & x - y + z & = & 0 \\ \hline & 0x + 0y + 0z & = & 3 \end{array}$$

and it is evident that the resulting equation can never be satisfied for any values of x, y, z.

4) The system

$$\begin{cases} x + 2y - z + 2w = 12 \\ y + z + 3w = 10 \\ x + 2y - 3z + 2w = 14 \\ 2x - 2y + 4z + 5w = 9 \end{cases}$$

has four equations and four variables. Its matrix equation is

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & -3 & 2 \\ 2 & -2 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 14 \\ 9 \end{pmatrix}.$$

It can be checked that the <u>only</u> solution is

$$(1, 2, -1, 3) \text{ or equivalently } \begin{cases} x = 1 \\ y = 2 \\ z = -1 \\ w = 3 \end{cases}.$$

5) Let us try to solve the system in the variables x and y,

$$\{x + y = 3\ .$$

Although this is an extremely simple system, it presents the problem that there are two variables but only one equation. This makes it impossible to calculate the value of one variable if we do not know the value of the other. If we intend to calculate x, we must know the value of y. So for example,

$$
\begin{array}{lll}
\bullet & \text{if } y=1 \text{ then} & x=3-y=3-1=2, \\
 & & \Downarrow \\
 & & x=2, y=1 \text{ is a solution,} \\
\hline
\bullet & \text{if } y=-7 \text{ then} & x=3-y=3-(-7)=10, \\
 & & \Downarrow \\
 & & x=10, y=-7 \text{ is a solution,} \\
\hline
\bullet & \text{if } y=0 \text{ then} & x=3-y=3-0=3, \\
 & & \Downarrow \\
 & & x=3, y=0 \text{ is a solution,} \\
 & \text{etc.} &
\end{array}
$$

As we see, the only way to find solutions of the system is to assume that we know the value of y. Instead of giving different values, generically we can assume that y takes a certain value α (which in the previous examples was $\alpha = 1$, $\alpha = -7$ or $\alpha = 0$) so that we have the system

$$\left\{ \begin{array}{l} x+y=3 \\ y=\alpha \end{array} \right.$$

whose solution is evidently

$$\left\{ \begin{array}{l} x=3-\alpha \\ y=\alpha \end{array} \right. \quad \text{or in another way} \quad (3-\alpha, \alpha).$$

As we see in this example, when we have few equations we do not have enough information to calculate the solutions of the system. In such a case we can proceed as here by assuming that we know the value of some of the variables and that this value is equal to a certain parameter (in this case α). It is often said then that we have 'taken a variable as a parameter'. Specifically, here we have taken the variable y as a parameter to solve the system.

By giving values to the parameter α we obtain all the solutions of the system. Thus we have,

$$
(3-\alpha,\alpha)
\begin{array}{l}
\xrightarrow[\alpha=1]{} (2,1) \text{ or in another way} \left\{ \begin{array}{l} x=2 \\ y=1 \end{array} \right. , \\
\xrightarrow[\alpha=-7]{} (10,-7) \text{ or in another way} \left\{ \begin{array}{l} x=10 \\ y=-7 \end{array} \right. , \\
\xrightarrow[\alpha=0]{} (3,0) \text{ or in another way} \left\{ \begin{array}{l} x=3 \\ y=0 \end{array} \right. , \\
\xrightarrow[\alpha=3]{} (0,3) \text{ or in another way} \left\{ \begin{array}{l} x=0 \\ y=3 \end{array} \right.
\end{array}
$$

and in general we will have infinitely many solutions corresponding to all other possible values of α. The set of all solutions will be

$$\{(3-\alpha, \alpha) : \alpha \in \mathbb{R}\}.$$

6) In the system,

$$\left\{ \begin{array}{l} x+y+z+w=2 \\ x-y+z-2w=1 \end{array} \right. ,$$

there are four variables and only two equations. With only two equations we cannot calculate the value of the four variables x, y, z, and w. As in the previous example, we will only be able to solve the system if we know the value of some of the variables. Let us suppose therefore that the value of z is α and that of w is β. That is, we have taken the variables z and w as parameters so the system remains in the form,

$$\left\{ \begin{array}{l} x+y+z+w=2 \\ x-y+z-2w=1 \\ z=\alpha \\ w=\beta \end{array} \right. .$$

Since we have assumed we know the variables z and w, we substitute them by their values and leave on the left-hand side of each equality only the variables we still do not know,

$$\left\{\begin{array}{l} x+y=2-\alpha-\beta \\ x-y=1-\alpha+2\beta \\ z=\alpha \\ w=\beta \end{array}\right. .$$

In this way, we are left to solve the system

$$\left\{\begin{array}{l} x+y=2-\alpha-\beta \\ x-y=1-\alpha+2\beta \end{array}\right. .$$

which has two variables and two equations and we will be able to find its solutions by giving a specific value for x and y which, of course, will appear as a function of the parameters α and β. To do this it will suffice to add or subtract the two equations from each other as follows:

$$\left\{\begin{array}{l} x+y=2-\alpha-\beta \\ x-y=1-\alpha+2\beta \end{array}\right. \quad \begin{array}{l} \xrightarrow[\text{adding the equations}]{} \quad 2x=3-2\alpha+\beta \\ \xrightarrow[\text{subtracting the equations}]{} \quad 2y=1-3\beta \end{array}$$

and the final solution is

$$\left\{\begin{array}{l} x=\frac{3-2\alpha+\beta}{2} \\ y=\frac{1-3\beta}{2} \\ z=\alpha \\ w=\beta \end{array}\right. \quad \text{or in another form, } (\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta).$$

The system has infinitely many solutions, all of which are obtained by giving values to the parameters α and β. For example,

$$\begin{array}{lcl} \alpha=1, \beta=0 & \rightarrow & (\frac{1}{2}, \frac{1}{2}, 1, 0) \\ \alpha=1, \beta=1 & \rightarrow & (1, -1, 1, 1) \end{array}$$

the other solutions can be obtained in this way. Thus, the set of all solutions of the system is

$$\{(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta) \in \mathbb{R}^4 : \alpha, \beta \in \mathbb{R}\}.$$

In the last two examples we have been able to write all the solutions of each system by taking variables as parameters. In example **5)** we took the variable y as a parameter assigning it the parameter α and in example **6)** we took the variables z and w as parameters assigning them the parameters α and β.

When a system has a unique solution, as in example **4)**, it is not necessary to use any parameter to solve it. Only when a system has several solutions will we need to use parameters to express them. In the following definition we specify what we understand by 'taking a variable as a parameter' in a system.

Definición 4. Given the linear system with m equations and n ordered unknowns $(x_1, x_2, \dots, x_n)$,

$$\left\{\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m \end{array}\right. ,$$

the system

$$\left\{\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m \\ x_i=\alpha_i \end{array}\right. ,$$

resulting from adding to the initial system the equation $x_i=\alpha_i$, is said to have been obtained by taking the variable x_i as a parameter via the parameter α_i.

Of course, in a system it is possible to successively take different variables as parameters. The question is to know how many variables and which ones we have to take as parameters in order to manage to arrive at the solution of the system. We can find systems that need two, one, or no parameters to be solved. In the following definition we give a classification of linear systems based on the characteristics of their solution set.

Definición 5. A linear system is said to be:

- **consistent:** If it has at least one solution.
- **inconsistent:** If it has no solution.
- **undetermined:** If it has more than one solution.
- **determined:** If it has a unique solution.
- d**-dimensional solution:** If in the system d variables can be selected such that the resulting system upon taking them as parameters is consistent and determined. That is, if all its solutions can be expressed by taking d variables of the system as parameters.

Schematically we have:

$$\text{Linear system}\begin{cases}\begin{array}{l}\text{Consistent}\\ \text{(with solution)}\end{array}\begin{cases}\begin{array}{l}\text{Determined}\\ \text{(unique solution)}\end{array}\\ \begin{array}{l}\text{Undetermined}\\ \text{(several solutions)}\end{array}\begin{cases}\text{1-dimensional solution.}\\ \text{2-dimensional solution.}\\ \quad\vdots\\ d\text{-dimensional solution.}\end{cases}\end{cases}\\ \begin{array}{l}\text{Inconsistent}\\ \text{(no solution)}\end{array}\end{cases}$$

We have already commented that a consistent system that has a unique solution does not need any parameter to be solved. That is, it needs 0 parameters and will consequently be a 0-dimensional solution system.

It is evident that a homogeneous system will always have at least one solution consisting of choosing all variables equal to zero. Therefore, every homogeneous system is always consistent. In case the system is non-homogeneous, it will be necessary to resort to more complex arguments to determine whether the system is consistent or not.

Ejemplo 6. Consider the system

$$\begin{cases} x+y=0\\ x+y=1\end{cases}.$$

It is evident that it has no solution since it is not possible to find any values of x and y that satisfy both equations at the same time (if x and y sum to 0, it is impossible for them to also sum to 1 at the same time). The system is therefore inconsistent.

In this case it was possible to see at a glance that the system is inconsistent. However, we may encounter other not so simple cases where it will not be so easy to determine whether the system in question is consistent or inconsistent.

The following theorem allows us to determine the type of a system based on the study of the ranks of its coefficient matrix and its augmented matrix.

Teorema 7 (Rouché-Frobenius)**.** *Consider a linear system with m equations and n unknowns expressed by its matrix form:*

$$A \cdot X = B,$$

where $A \in \mathcal{M}_{m\times n}$*,* $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ *and* $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$*. Then:*

i) The system is consistent $\Leftrightarrow \operatorname{rango}(A) = \operatorname{rango}(A|B)$*.*

ii) The system is determined $\Leftrightarrow \operatorname{rango}(A) = \operatorname{rango}(A|B) = n =$ *number of unknowns.*

iii) The system has a d-dimensional solution ($d > 0$) $\Leftrightarrow n - \operatorname{rango}(A) = d$*.*

Proof. Let us now only see the proof of point *i)*. The proof of points *ii)* and *iii)* is easily achieved by resorting to the arguments we will use later in **Property 17**.

If we consider the m-tuple columns of A, $v_1, v_2, \ldots, v_n$ we have that $A = (v_1|v_2|\cdots|v_n)$ and the system can be written in the form

$$(v_1|v_2|\cdots|v_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = B.$$

Taking into account the representations seen in Chapter 4 (page **??**) for the product of a matrix of tuple columns by a column, we can write this last equality in the form

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = B.$$

In such a case, the system will have a solution if there exist numbers $x_1, x_2, \ldots, x_n$ that, when substituted into this expression, make it true. But if the last equality holds then B can be obtained as a linear combination of $v_1, v_2, \ldots, v_n$. In summary,

$$\text{The system has a solution} \Leftrightarrow B \in \langle v_1, v_2, \ldots, v_n\rangle \Leftrightarrow \operatorname{rango}(v_1|v_2|\cdots|v_n) = \operatorname{rango}(v_1|v_2|\cdots|v_n|B)$$

$$\Leftrightarrow \operatorname{rango}(A) = \operatorname{rango}(A|B),$$

where the last implication is a consequence of **Property 152** of Chapter 4. □

In a system with a d-dimensional solution, $d > 0$, the set of solutions depends on certain parameters that can take any value and therefore the system will have infinitely many solutions. A consistent system is always either determined with a unique solution or undetermined and therefore of d-dimensional solution, $d > 0$, with infinitely many solutions. Consequently, a system has either no solution (inconsistent system) or a unique solution (determined) or infinitely many solutions (undetermined)

Ejemplos 8.

1) The system

$$\begin{cases} 2x - y + z + w = 3 \\ y - z + w = 2 \\ 2x + z + w = -1 \\ 2x + 3y + z + 2w = 4 \end{cases}$$

has coefficient and augmented matrices equal to

$$A = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad (A|B) = \left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 2 \\ 2 & 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & 2 & 4 \end{array}\right).$$

We have

$$\text{rango}(A) = 4, \quad \text{rango}(A|B) = 4.$$

Therefore

$$\text{rango}(A) = \text{rango}(A|B) = 4 = \text{no. of variables}$$

and the system is consistent and determined.

2) The system

$$\begin{cases} 2x - y + 2z + w = 3 \\ x + 2z + w = 3 \\ y + 2z + w = 3 \\ x - 2y - 2z - w = -3 \end{cases}$$

has coefficient and augmented matrices equal to

$$A = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & -2 & -2 & -1 \end{pmatrix} \quad \text{and} \quad (A|B) = \begin{pmatrix} 2 & -1 & 2 & 1 & 3 \\ 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -2 & -2 & -1 & -3 \end{pmatrix}.$$

We have

$$\text{rango}(A) = 2, \quad \text{rango}(A|B) = 2.$$

Therefore

$$\text{rango}(A) = \text{rango}(A|B) = 2 < \text{no. of variables}$$

and the system is consistent and undetermined with a

$$n - \text{rango}(A) = 4 - 2 = 2\text{-dimensional solution.}$$

3) The system

$$\begin{cases} 2x - y + 2z + w = 3 \\ x + 2z + w = 3 \\ y + 2z + w = 3 \\ x - 2y - 2z - w = 7 \end{cases}$$

has coefficient and augmented matrices equal to

$$A = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & -2 & -2 & -1 \end{pmatrix} \quad \text{and} \quad (A|B) = \begin{pmatrix} 2 & -1 & 2 & 1 & 3 \\ 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -2 & -2 & -1 & 7 \end{pmatrix}.$$

We have

$$\text{rango}(A) = 2, \quad \text{rango}(A|B) = 3.$$

Therefore

$$\text{rango}(A) \neq \text{rango}(A|B)$$

and the system is inconsistent.

5.2 Gaussian elimination method

In this section we study a first technique for finding the solution of a linear system. This method is based on the use of appropriate elementary operations to progressively simplify the initial system until transforming it into another whose solution can be calculated immediately. These elementary operations will be applied to the detailed matrix of the system, which collects in a single block matrix both the augmented matrix and the column of variables. The following definition precisely describes what we understand by the detailed matrix of a system.

Definición 9. Given the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases},$$

we call the detailed matrix of the system the matrix

$$\text{numerical rows}\left\{\left(\begin{array}{cccc|c} x_1 & x_2 & \dots & x_n & 0 \\ \hline a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)\right. .$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{variable columns}}$$

Numerical rows of the detailed matrix are all its rows except the first one, and variable columns of the detailed matrix are those columns whose first element is one of the variables x_i.

Ejemplos 10.

1) The detailed matrix of the linear system $\begin{cases} x_1 + 2x_2 + x_3 + 6x_4 = 1 \\ 2x_1 + 4x_2 - x_3 + 3x_4 = 1 \\ -x_1 - 2x_2 + 2x_3 - x_4 = 2 \end{cases}$ is $\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline 1 & 2 & 1 & 6 & 1 \\ 2 & 4 & -1 & 3 & 1 \\ -1 & -2 & 2 & -1 & 2 \end{array}\right)$.

2) Corresponding to the detailed matrix $\left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right)$ is the system

$$\begin{cases} x + 2y + z = -1 \\ -x + 2y + 3z = 2 \end{cases}.$$

If we modify the detailed matrix we will also be modifying the corresponding system. Thus for example if we apply various modifications, the obtained systems are:

$$* \quad \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[\text{F2}\leftrightarrow\text{F3}]{} \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline -1 & 2 & 3 & 2 \\ 1 & 2 & 1 & -1 \end{array}\right) \Updownarrow \begin{cases} -x + 2y + 3z = 2 \\ x + 2y + z = -1 \end{cases}.$$

$$
* \quad \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[\text{C1}\leftrightarrow\text{ C2}]{} \left(\begin{array}{ccc|c} y & x & z & 0 \\ \hline 2 & 1 & 1 & -1 \\ 2 & -1 & 3 & 2 \end{array}\right) .
$$

$$
\Updownarrow
$$

$$
\begin{cases} 2y + x + z = -1 \\ 2y - x + 3z = 2 \end{cases}
$$

$$
* \quad \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[\text{2F2}]{} \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 2 & 4 & 2 & -2 \\ -1 & 2 & 3 & 2 \end{array}\right) .
$$

$$
\Updownarrow
$$

$$
\begin{cases} 2x + 4y + 2z = -2 \\ -x + 2y + 3z = 2 \end{cases}
$$

$$
* \quad \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ -1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[\text{F3=F3+F2}]{} \left(\begin{array}{ccc|c} x & y & z & 0 \\ \hline 1 & 2 & 1 & -1 \\ 0 & 4 & 4 & 1 \end{array}\right) .
$$

$$
\Updownarrow
$$

$$
\begin{cases} x + 2y + z = -1 \\ 4y + 4z = 1 \end{cases}
$$

By applying different transformations we have also obtained different linear systems, but it is easy, nevertheless, to check that all these systems have the same solutions and are therefore equivalent to the initial linear system. Thus we see that there exist certain transformations of the detailed matrix that do not alter the system that this matrix represents or that transform it into an equivalent one.

We will call these transformations of the detailed matrix that lead to equivalent systems with the same solutions, 'elementary operations for the detailed matrix'. We will consider those that appear in the following definition. As we have already said, the objective will be to transform the initial system into another whose detailed matrix is simpler.

Definición 11. We call an elementary operation for the detailed matrix of a system any of the following actions:

1. Modify the order of the numerical rows.
2. Modify the order of the variable columns.
3. Multiply a numerical row by a non-zero number.
4. Add to a numerical row another numerical row multiplied by a number.

As we have already seen, the detailed matrix of a system is divided into two blocks, one for the variable columns and one for the column of constant terms. To solve a system, it is interesting to obtain as many zeros as possible in the block corresponding to the variable columns of the system, and for this we will apply the Gaussian elimination method. Now the rules are similar to those we already indicated in the previous topic for calculating the inverse matrix. We must, therefore, iteratively follow these steps:

1. We select one of the variable columns.
2. In the selected column we choose a non-zero element (pivot), which must be at a height different from those selected in previous steps.
3. Using the pivot, we nullify the elements of the selected column.

In the procedure we will follow, in principle, it is not necessary to modify the order of the variable columns or the numerical rows.

We finish applying these three steps once we have reduced all the variable columns or there are no more non-zero pivots at heights different from those already done. Then we say that the system is in *reduced row echelon form.* A system in reduced row echelon form is solved immediately taking into account the following points:

- If after reducing the matrix through elementary operations a row appears with all its elements zero in the variable columns and the element in the constant term is non-zero, then the system will be inconsistent.

Ejemplo 12. In the following detailed matrix,

$$\left(\begin{array}{cccc|c} x & y & z & w & 0 \\ \hline 1 & 0 & -1 & 1 & -1 \\ 0 & -1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right),$$

a complete row of zeros appears accompanied in the constant terms by a non-zero element. If we write the equation corresponding to this row we would have

$$0x + 0y + 0z + 0w = 2$$

and it is evident that no valid solution exists for this equation. Therefore the system is inconsistent.

- We will solve for the variables corresponding to the columns we have reduced.
- We will take as parameters the variables corresponding to the columns that have not been reduced.

Ejemplo 13. In the following matrix,

$$\left(\begin{array}{cccc|c} \mathrm{x} & y & \mathrm{z} & w & 0 \\ \hline \underline{1} & 1 & 0 & 1 & -1 \\ 0 & -1 & \underline{2} & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

after reducing two columns we cannot choose more non-zero pivots at a height different from those already marked. Therefore we will not apply more elementary operations and the matrix is already in reduced row echelon form. The columns we have reduced correspond to the variables x and z and these will be the ones we solve for, the other two variables y, w, will be the ones we take as parameters. To solve, we simply rewrite (of course, it is not necessary to write the third equation since it is entirely zero) the system and solve as indicated:

$$\left\{\begin{array}{l} x+y+w=-1, \\ -y+2z+3w=2, \\ y=\alpha, \\ w=\beta \end{array}\right. \underset{\substack{\text{Substituting} \\ \text{parameters} \\ \text{and solving}}}{\Rightarrow} \left\{\begin{array}{l} x=-1-\alpha-\beta, \\ z=1+\frac{\alpha}{2}-\frac{3}{2}\beta, \\ y=\alpha, \\ w=\beta. \end{array}\right. \Rightarrow \left\{\begin{array}{l} x=-1-\alpha-\beta, \\ y=\alpha, \\ z=1+\frac{\alpha}{2}-\frac{3}{2}\beta, \\ w=\beta. \end{array}\right.$$

Let's see some examples where we reproduce this technique for solving systems.

Ejemplos 14.
1) Let's solve the system

$$\begin{cases} x_1 + x_2 - 2x_3 + 2x_4 + x_5 = 3 \\ 5x_1 + 3x_2 - 3x_3 + 4x_4 + 2x_5 = 4 \\ 3x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 1 \\ 2x_1 + 4x_2 - 2x_3 + x_4 = 1 \\ 3x_1 + 4x_2 - 2x_3 + x_4 = 1 \end{cases}.$$

The detailed matrix of the system is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 2 & 1 & 3 \\ 5 & 3 & -3 & 4 & 2 & 4 \\ 3 & 2 & -1 & 2 & 1 & 1 \\ 2 & 4 & -2 & 1 & 0 & 1 \\ 2 & 4 & -2 & 1 & 0 & 1 \end{array}\right) \end{array}.$$

Let's obtain the reduced row echelon matrix. To do this, we will take different pivotes, all at different heights, to nullify each column.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 2 & 1 & 3 \\ 5 & 3 & -3 & 4 & 2 & 4 \\ 3 & 2 & -1 & 2 & 1 & 1 \\ 2 & 4 & -2 & 1 & 0 & 1 \\ 2 & 4 & -2 & 1 & 0 & 1 \end{array}\right) \end{array} \xrightarrow[\substack{\text{F3 = F3 - 5F2} \\ \text{F4 = F4 - 3F2} \\ \text{F5 = F5 - 2F2} \\ \text{F6 = F6 - 3F2}}]{\text{(pivot=2}^{\text{nd}}\text{ element of the 1}^{\text{st}}\text{ column)}} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} \underline{\mathbf{1}} & 1 & -2 & 2 & 1 & 3 \\ 0 & 2 & -7 & -6 & -3 & -11 \\ 0 & -1 & 5 & -4 & -2 & -8 \\ 0 & 2 & 2 & -3 & -2 & -5 \\ 0 & 1 & 4 & -5 & -3 & -8 \end{array}\right) \end{array}$$

$$\xrightarrow[\substack{\text{F2 = F2 - F6} \\ \text{F3 = F3 + 2F6} \\ \text{F4 = F4 + F6} \\ \text{F5 = F5 - 2F6}}]{\text{(pivot=6}^{\text{th}}\text{ element of the 2}^{\text{nd}}\text{ column)}} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} \underline{\mathbf{1}} & 0 & -6 & 7 & 4 & 11 \\ 0 & 0 & 15 & -16 & -9 & -27 \\ 0 & 0 & 9 & -9 & -5 & -16 \\ 0 & 0 & -6 & 7 & 4 & 11 \\ 0 & \underline{\mathbf{1}} & 4 & -5 & -3 & -8 \end{array}\right) \end{array}$$

$$\xrightarrow[\substack{\text{F2 = F2 + }\frac{6}{9}\text{ F4} \\ \text{F3 = F3 - }\frac{15}{9}\text{ F4} \\ \text{F5 = F5 + }\frac{6}{9}\text{F4} \\ \text{F6 = F6 - }\frac{4}{9}\text{F4}}]{\text{(pivot=4}^{\text{th}}\text{ element of the 3}^{\text{rd}}\text{ column)}} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} \underline{\mathbf{1}} & 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \underline{\mathbf{9}} & -9 & -5 & -16 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & \underline{\mathbf{1}} & 0 & -1 & -\frac{7}{9} & -\frac{8}{9} \end{array}\right) \end{array}$$

$$\xrightarrow[\substack{\text{F2 = F2 + F3} \\ \text{F4 = F4 - 9F3} \\ \text{F5 = F5 + F3} \\ \text{F6 = F6 - F3}}]{\text{(pivot=3}^{\text{rd}}\text{ element of the 4}^{\text{th}}\text{ column)}} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{array} \\ \left(\begin{array}{ccccc|c} \underline{\mathbf{1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{\mathbf{-1}} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \underline{\mathbf{9}} & 0 & 1 & -13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \underline{\mathbf{1}} & 0 & 0 & -\frac{1}{9} & -\frac{5}{9} \end{array}\right) \end{array}$$

It can be observed that it is no longer possible to choose any non-zero pivot at a height different from those previously selected. This indicates that we already have the reduced row echelon form. The simple observation of it allows us to affirm that it is a consistent and undetermined system with a 1-dimensional solution that needs a single parameter to be solved. The variables corresponding to the columns we have reduced are x_1, x_2, x_3, x_4 and these will be the ones we solve for, and therefore we must take the variable x_5 as a parameter to obtain the solution. The advantage of the reduced row echelon form lies in the fact that, once we reach it, calculating the solutions does not require more operations. Indeed, if we write the system corresponding to the reduced row echelon matrix and take the variable x_5 as a parameter we have,

$$\begin{cases} x_1 = 0 \\ -x_4 - \frac{2}{3}x_5 = -\frac{1}{3} \\ 9x_3 + x_5 = -13 \\ 0 = 0 \\ x_2 - \frac{1}{9}x_5 = -\frac{5}{9} \\ x_5 = \alpha \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_4 = \frac{1}{3} - \frac{2}{3}\alpha \\ x_3 = -\frac{13}{9} - \frac{\alpha}{9} \\ x_2 = -\frac{5}{9} + \frac{1}{9}\alpha \\ x_5 = \alpha \end{cases}.$$

From which we directly deduce that the solution of the system is

$$\begin{cases} x_1 = 0 \\ x_2 = -\frac{5}{9} + \frac{1}{9}\alpha \\ x_3 = -\frac{13}{9} - \frac{1}{9}\alpha \\ x_4 = \frac{1}{3} - \frac{2}{3}\alpha \\ x_5 = \alpha \end{cases} \quad \text{or in the form} \quad (0, -\frac{5}{9} + \frac{1}{9}\alpha, -\frac{13}{9} - \frac{1}{9}\alpha, \frac{1}{3} - \frac{2}{3}\alpha, \alpha)$$

as a function of the parameter α.

2) Let us now study the system

$$\begin{cases} 2x_1 - x_2 + 3x_3 + 2x_4 = 1 \\ 3x_1 - 2x_2 + 4x_3 = 2 \\ x_1 + 2x_2 + x_4 = 1 \\ 3x_1 + 3x_2 + x_3 = 2 \\ 2x_1 + x_2 + x_3 - x_4 = 2 \end{cases}.$$

To do this we will try to reach its row echelon form. To do this we will select pivots and nullify the corresponding columns.

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline 2 & -1 & 3 & 2 & 1 \\ 3 & -2 & 4 & 0 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 3 & 1 & 0 & 2 \\ 2 & 1 & 1 & -1 & 2 \end{array}\right) \xrightarrow[\substack{\text{F2 = F2 - 2F4} \\ \text{F6 = F6 + F4}}]{\text{(pivot=4}^{\text{th}}\text{ element of the 4}^{\text{th}}\text{ column)}} \left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline 0 & -5 & 3 & 0 & -1 \\ 3 & -2 & 4 & 0 & 2 \\ 1 & 2 & 0 & \underline{\overline{\mathbf{1}}} & 1 \\ 3 & 3 & 1 & 0 & 2 \\ 3 & 3 & 1 & 0 & 3 \end{array}\right).$$

$$\xrightarrow[\substack{\text{F2 = F2 - 3F5} \\ \text{F3 = F3 -4F5} \\ \text{F6 = F6 -F5}}]{\text{(pivot=5}^{\text{th}}\text{ element of the 3}^{\text{rd}}\text{ column)}} \left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & 0 \\ \hline -9 & -14 & 0 & 0 & -7 \\ -9 & -16 & 0 & 0 & -6 \\ 1 & 2 & 0 & \underline{\overline{\mathbf{1}}} & 1 \\ 3 & 3 & \underline{\overline{\mathbf{1}}} & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

We could actually continue reducing columns but we observe that a row appears, the last one, all zeros accompanied by a non-zero constant term. Without needing to reach reduced row echelon form we deduce that the system is inconsistent.

5.3 Cramer's Rule (Solving systems via inverse matrix calculation)

Given any system we can always extract its matrix equation which will be of the form

$$AX = B, \qquad A \in \mathcal{M}_{m\times n}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then, it is possible to apply the rules for manipulating matrix equalities from Chapter 4 to solve the system. In fact, it would suffice to solve for X in that matrix equation to arrive at the solution in the form

$$X = A^{-1} \cdot B.$$

This last point provides us with a valid method for solving systems. However, we must bear in mind that solving in this way is feasible only when the matrix A is square (or equivalently, the system has as many equations as variables) and regular. A system that satisfies these conditions and that can be solved by solving for the coefficient matrix is called a 'Cramer's system'. More precisely we have:

Propiedad 15. *Consider the system with n equations and variables $(x_1, x_2, \dots, x_n)$, given by its matrix equation*

$$A \cdot X = B,$$

where

$$A = (a_{ij})_{n\times n}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then, if $|A| \neq 0$ the system is consistent and determined and its solution is

$$X = A^{-1} \cdot B = \frac{1}{|A|}\operatorname{Adj}(A)^t \cdot B. \tag{5.1}$$

The last expression in (5.1) is what is usually known as Cramer's rule. Generally, Cramer's rule is presented by developing the matrix product indicated in the property, solving for each variable individually. It then takes the following formulation:

$$x_i = \frac{\begin{vmatrix} a_{1,1} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{n,n} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}}, \qquad i = 1, \dots, n.$$

Note that in these formulas, for the calculation of the i-th variable, the denominator contains the determinant of the coefficient matrix and the numerator contains the determinant of the matrix resulting from replacing the i-th column of the coefficient matrix with the constant terms of the system.

Ejemplo 16. Let's try to solve the system

$$\begin{cases} 3x + 2y - z = 9, \\ 2x + y + z = 2, \\ x + 2y + 2z = -1. \end{cases}$$

using the previous ideas. First we must verify that the system is indeed a Cramer system, i.e., that it has the same number of equations as variables, which is evident, and that the determinant of the coefficient matrix is different from zero. So we begin by calculating the determinant of the coefficient matrix:

$$\left|\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right| = -9 \neq 0.$$

Since the determinant is non-zero, the system is indeed a Cramer system. Now we can follow two paths to solve the system. First, we could use matrix calculation as it appears in equation (5.1) and, second, it would be possible to resort to the Cramer's rule equations. Let's see how we would proceed in both cases.

Method 1: Via matrix calculation. We write the system using its matrix expression,

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}.$$

Since the coefficient matrix is square and has a non-zero determinant, we can calculate its inverse and solve for it in the last equality as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}.$$

We can calculate the inverse of the coefficient matrix using any of the methods we know for this. In this case, we obtain

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}^{-1} = \frac{1}{9}\begin{pmatrix} 0 & 6 & -3 \\ 3 & -7 & 5 \\ -3 & 4 & 1 \end{pmatrix}.$$

Now we substitute the inverse by its value and perform the matrix products that appear to obtain the final result

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 0 & 6 & -3 \\ 3 & -7 & 5 \\ -3 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 15 \\ 8 \\ -20 \end{pmatrix} = \begin{pmatrix} \frac{15}{9} \\ \frac{8}{9} \\ \frac{-20}{9} \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{8}{9} \\ \frac{-20}{9} \end{pmatrix} \Rightarrow \begin{cases} x = \frac{5}{3}, \\ y = \frac{8}{9}, \\ z = \frac{-20}{9}. \end{cases}$$

Method 2: Via Cramer's rule. The Cramer's formulas for solving this system would be the following:

$$x = \frac{\left|\begin{pmatrix} \mathbf{9} & 2 & -1 \\ \mathbf{2} & 1 & 1 \\ \mathbf{-1} & 2 & 2 \end{pmatrix}\right|}{\left|\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right|}, \qquad y = \frac{\left|\begin{pmatrix} 3 & \mathbf{9} & -1 \\ 2 & \mathbf{2} & 1 \\ 1 & \mathbf{-1} & 2 \end{pmatrix}\right|}{\left|\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right|}, \qquad z = \frac{\left|\begin{pmatrix} 3 & 2 & \mathbf{9} \\ 2 & 1 & \mathbf{2} \\ 1 & 2 & \mathbf{-1} \end{pmatrix}\right|}{\left|\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}\right|}.$$

Note that, in the numerator, for the first variable we replace the first column of the coefficient matrix with the constant terms of the system, for the second variable we replace the second column and for the third variable the third column. Since we have already calculated the determinant of the coefficient matrix, we will only have to solve the three determinants that appear in the numerator. We have that

$$\left|\begin{pmatrix} \mathbf{9} & 2 & -1 \\ \mathbf{2} & 1 & 1 \\ \mathbf{-1} & 2 & 2 \end{pmatrix}\right| = -15, \qquad \left|\begin{pmatrix} 3 & \mathbf{9} & -1 \\ 2 & \mathbf{2} & 1 \\ 1 & \mathbf{-1} & 2 \end{pmatrix}\right| = -8, \qquad \left|\begin{pmatrix} 3 & 2 & \mathbf{9} \\ 2 & 1 & \mathbf{2} \\ 1 & 2 & \mathbf{-1} \end{pmatrix}\right| = 20,$$

so finally

$$x = \frac{-15}{-9} = \frac{5}{3}, \qquad y = \frac{-8}{-9} = \frac{8}{9}, \qquad z = \frac{20}{-9} = -\frac{20}{9}.$$

We thus have two alternative ways to solve a Cramer system. In principle, for systems with two, three, or four equations, both methods present an equivalent difficulty and both, the one based on matrix operations and the one using Cramer's rule, can be used interchangeably.

One might ask whether it is possible to modify this technique in some way to allow its use for any type of system and not only for Cramer systems. The key to answering this question is found in the alternative definition of the rank of a matrix that we gave in **Property 162**. There we saw that the rank is the order of the largest regular minor of a matrix. If the matrix A is not regular or is not square we will not be able to obtain its inverse but if we have that $\text{rango}(A) = r$ we can find within A a submatrix that is regular. Although the initial system may not have the same number of variables as equations, once said regular submatrix is detected we can modify this situation if we take into account that:

- ⋆ In a system we can eliminate those equations that are superfluous.
- ⋆ The number of equations can be modified by taking variables as parameters.

In the following property we develop in detail the idea of the previous paragraph that will allow extending Cramer's method to any system.

Propiedad 17. *Consider the system with m equations and variables $(x_1, x_2, \dots, x_n)$, given by its matrix equation*

$$A \cdot X = B,$$

where

$$A = (a_{ij})_{m\times n}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \textit{and} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Suppose the system is consistent, in which case

$$\text{rango}(A \mid B) = \text{rango}(A) = r.$$

Consider the minor $\tilde{A}_{r\times r}$ of A obtained as the intersection of rows $i_1, i_2, \dots, i_r$ and columns $j_1, j_2, \dots, j_r$ of A and suppose that $|\tilde{A}| \neq 0$. Then:

i) The system composed solely of the i_1-th, i_2-th, ..., i_r-th equations of the initial system has the same solutions as it.

ii) The system obtained by taking as a parameter any variable that is not one of $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ is determined.

In other words, once we have detected within the coefficient matrix, A, a regular minor of order r, we must eliminate all equations that do not participate in the minor and take as parameters all variables whose column does not participate in the minor.

Ejemplo 18. In principle, to solve the system

$$\begin{cases} x_1 + 2x_2 - 5x_3 - x_4 = -1 \\ 2x_1 + 3x_2 - 7x_3 + x_4 = 2 \\ x_1 + x_2 - 2x_3 + 2x_4 = 3 \\ 2x_1 + x_2 - x_3 + 6x_4 = 9 \\ 3x_1 + 7x_2 - 18x_3 - 5x_4 = -6 \end{cases}$$

it would not be possible to apply Cramer's method. However we can resort to **Property 17** to transform the system appropriately. First, since

$$\text{rango}\begin{pmatrix} 1 & 2 & -5 & -1 \\ 2 & 3 & -7 & 1 \\ 1 & 1 & -2 & 2 \\ 2 & 1 & -1 & 6 \\ 3 & 7 & -18 & -5 \end{pmatrix} = \text{rango}\left(\begin{array}{cccc|c} 1 & 2 & -5 & -1 & -1 \\ 2 & 3 & -7 & 1 & 2 \\ 1 & 1 & -2 & 2 & 3 \\ 2 & 1 & -1 & 6 & 9 \\ 3 & 7 & -18 & -5 & -6 \end{array}\right) = 3$$

the system is consistent and undetermined with a 1-dimensional solution. The rank of the coefficient matrix is equal to 3 and therefore there will exist within it a regular submatrix of order 3. If we take the submatrix corresponding to the first, second and fourth rows and to the first, second and fourth columns we have that

$$\text{rango}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix} = 3$$

and therefore said minor is regular, it has an inverse. Let's mark in the coefficient matrix of the system the rows and columns corresponding to the selected minor,

$$\begin{array}{r c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{r} \text{equation } 1^{\text{st}} \\ \text{equation } 2^{\text{nd}} \\ \text{equation } 3^{\text{rd}} \\ \text{equation } 4^{\text{th}} \\ \text{equation } 5^{\text{th}} \end{array} & \begin{pmatrix} \mathbf{1} & \mathbf{2} & -5 & \mathbf{-1} \\ \mathbf{2} & \mathbf{3} & -7 & \mathbf{1} \\ 1 & 1 & -2 & 2 \\ \mathbf{2} & \mathbf{1} & -1 & \mathbf{6} \\ 3 & 7 & -18 & -5 \end{pmatrix} \end{array}.$$

Let's eliminate the equations that do not participate in the minor and take as a parameter the variables that also do not. After this, the system becomes

$$\left\{\begin{array}{l} x_1 + 2x_2 - x_4 = 5\alpha - 1 \\ 2x_1 + 3x_2 + x_4 = 7\alpha + 2 \\ 2x_1 + x_2 + 6x_4 = \alpha + 9 \\ x_3 = \alpha \end{array}\right. .$$

Since, as a function of the parameter α, we already know the value of x_3, we will solve only the system

$$\left\{\begin{array}{l} x_1 + 2x_2 - x_4 = 5\alpha - 1 \\ 2x_1 + 3x_2 + x_4 = 7\alpha + 2 \\ 2x_1 + x_2 + 6x_4 = \alpha + 9 \end{array}\right. .$$

But now we have the same number of equations and variables and furthermore the coefficient matrix is precisely the minor selected before which is regular. Therefore this system is a Cramer system and can be solved by solving in the matrix equation which is

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix}.$$

Solving and calculating the inverse, we finally have

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix} = \begin{pmatrix} 17 & -13 & 5 \\ -10 & 8 & -3 \\ -4 & 3 & -1 \end{pmatrix} \begin{pmatrix} 5\alpha - 1 \\ 7\alpha + 2 \\ \alpha + 9 \end{pmatrix} = \begin{pmatrix} 2 - \alpha \\ 3\alpha - 1 \\ 1 \end{pmatrix}.$$

The solution of the system will then be,

$$\left\{\begin{array}{l} x_1 = 2 - \alpha, \\ x_2 = 3\alpha - 1, \\ x_3 = \alpha, \\ x_4 = 1 \end{array}\right. \quad \text{or equivalently } (2 - \alpha, 3\alpha - 1, \alpha, 1).$$

5.4 Expressing the solution of a system using linear combinations

We have seen that to express the solutions of a system of linear equations we need to introduce parameters. The representation of the solution of the system that we obtain through them is called the 'parametric expression' of the solution of the system. However, in this section we will see that we can also describe the solutions of a system using linear combinations. To obtain this representation via linear combinations we will take as a basis the parametric expression of the solution of the system. From its tuple form, it will suffice to appropriately separate the tuples corresponding to each parameter. We will only need to introduce the following notation:

Definición 19. Given a subset of tuples $C \subseteq \mathbb{R}^n$ and a fixed tuple $p \in \mathbb{R}^n$, the set of tuples obtained by adding the fixed tuple, p, to all tuples of C, is denoted $p + C$. That is:

$$p + C = \{p + c : c \in C\}.$$

Ejemplo 20. Consider the following set of 2-tuples

$$C = \{(1,0),(2,3),(-1,4)\} \subseteq \mathbb{R}^2$$

and consider the fixed tuple $(2,-1) \in \mathbb{R}^2$. Then the set $p+C$ is the one obtained by adding the tuple $(2,-1)$ to all those in C:

$$\begin{aligned} p+C &= (2,-1)+\{(1,0),(2,3),(-1,4)\} \\ &= \{(2,-1)+(1,0),(2,-1)+(2,3),(2,-1)+(-1,4)\} \\ &= \{(3,-1),(4,2),(1,3)\}. \end{aligned}$$

Let's now see through some examples how we can obtain the expression of the solution of a system using linear combinations:

Ejemplos 21.
1) Consider the linear system,

$$S \equiv \begin{cases} x+y+z+w=2 \\ x-y+z-2w=1 \end{cases}.$$

In part **3)** of **Examples 3** (page 193) we solved this system by taking two variables as parameters, so it is a system with a 2-dimensional solution. Using the parameters α and β the solution is written in the form

$$\begin{cases} x = \frac{3-2\alpha+\beta}{2}, \\ y = \frac{1-3\beta}{2}, \\ z=\alpha, \\ w=\beta \end{cases} \quad \text{or equivalently} \quad (\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta).$$

These last two representations are what are called parametric expressions of the solution of the system. To express the solutions using linear combinations we will use the tuple form (the second one) of the parametric expression of the solution. We will begin by separating in each component the terms that correspond to each of the two parameters and those that do not correspond to any of them, and then proceed as indicated below:

$$(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta) = (\underbrace{\frac{3}{2}-\alpha+\frac{1}{2}\beta}_{\substack{\text{Part without parameters: } \frac{3}{2} \\ \text{Part for } \alpha: -\alpha \\ \text{Part for } \beta: \frac{1}{2}\beta}}, \underbrace{\frac{1}{2}-\frac{3}{2}\beta}_{\substack{\text{Part without parameters: } \frac{1}{2} \\ \text{Part for } \alpha: 0 \\ \text{Part for } \beta: -\frac{3}{2}\beta}}, \underbrace{\alpha}_{\substack{\text{Part without parameters: } 0 \\ \text{Part for } \alpha: \alpha \\ \text{Part for } \beta: 0}}, \underbrace{\beta}_{\substack{\text{Part without parameters: } 0 \\ \text{Part for } \alpha: 0 \\ \text{Part for } \beta: \beta}})$$

$$\left(\text{We separate into different tuples the part that does not correspond to any parameter and those that correspond to each parameter}\right) = \underbrace{(\frac{3}{2}, \frac{1}{3}, 0, 0)}_{\text{Part without parameters}} + \underbrace{(-\alpha, 0, \alpha, 0)}_{\text{Part for } \alpha} + \underbrace{(\frac{1}{2}\beta, -\frac{3}{2}\beta, 0, 1)}_{\text{Part for } \beta}$$

$$\left(\text{We factor out the common term in the tuple of each parameter}\right) = (\frac{3}{2}, \frac{1}{3}, 0, 0) + \alpha(-1,0,1,0) + \beta(\frac{1}{2}, -\frac{3}{2}, 0, 1)$$

In summary we have that

$$(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta) = \underbrace{(\frac{3}{2}, \frac{1}{3}, 0, 0)}_{\text{Fixed tuple}} + \underbrace{\alpha(-1,0,1,0)+\beta(\frac{1}{2}, -\frac{3}{2}, 0, 1)}_{\substack{\text{Linear combination of} \\ (-1,0,1,0) \text{ and} \\ (\frac{1}{2}, -\frac{3}{2}, 0, 1)}}$$

We observe then that all solutions of the system are obtained by adding the fixed tuple $(\frac{3}{2}, \frac{1}{3}, 0, 0)$ to a linear combination of the tuples $(-1, 0, 1, 0)$ and $(\frac{1}{2}, -\frac{3}{2}, 0, 1)$. Now, the set of solutions of the system is obtained by giving different values to the parameters α and β, so the solution of the system can be expressed as

$$\{\underbrace{(\frac{3-2\alpha+\beta}{2}, \frac{1-3\beta}{2}, \alpha, \beta)}_{(\frac{3}{2},\frac{1}{3},0,0) \text{ plus an element of } \langle(-1,0,1,0),(\frac{1}{2},-\frac{3}{2},0,1)\rangle} : \alpha, \beta \in \mathbb{R}\} = (\frac{3}{2}, \frac{1}{3}, 0, 0) + \langle(-1, 0, 1, 0), (\frac{1}{2}, -\frac{3}{2}, 0, 1)\rangle.$$

This last expression,

$$(\frac{3}{2}, \frac{1}{3}, 0, 0) + \langle(-1, 0, 1, 0), (\frac{1}{2}, -\frac{3}{2}, 0, 1)\rangle,$$

is the representation of the set of solutions of the system using linear combinations.

2) Let's calculate the solution of the following system expressing it in its parametric form and using linear combinations,

$$H \equiv \left\{ \begin{array}{l} x_1 + x_2 - 2x_3 + 2x_4 + x_5 = 0 \\ 5x_1 + 3x_2 - 3x_3 + 4x_4 + 2x_5 = 0 \\ 3x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 0 \\ 2x_1 + 4x_2 - 2x_3 + x_4 = 0 \\ 3x_1 + 4x_2 - 2x_3 + x_4 = 0 \end{array} \right. .$$

Again we begin by solving the system. We can do this using any of the techniques from Chapter 5, obtaining as a result that the system has a 1-dimensional solution. As a function of the parameter α, all solutions of the system are written as

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha).$$

This would be the solution of the system expressed in parametric form. Starting from it, separating the tuples corresponding to each parameter (in this case we have a single parameter α) and the terms without a parameter, we have,

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha) = (0, 0, 0, 0, 0) + \alpha(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1).$$

Therefore, the expression of the solution of the system using linear combinations is

$$(0, 0, 0, 0, 0) + \langle(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1)\rangle$$

or, equivalently,

$$\langle(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1)\rangle.$$

Note that when the system is homogeneous the fixed tuple that appears in the representation using linear combinations is the null tuple and we can easily eliminate it. However, this does not happen in the case of non-homogeneous systems for which the fixed tuple will never be null.

5.5 Vector subspaces of $\mathbb{R}^n$

In the last example of the previous section, we managed to express the set of solutions of the homogeneous linear system that appeared as the set of linear combinations of a set of vectors. In this section we are going to prove that this property is true for any homogeneous linear system and important properties will be derived from it that we will take advantage of in later chapters (for example, in the diagonalization of matrices).

5.5.1 Definition and properties

Given a homogeneous linear system of equations with variables $(x_1, x_2, \ldots, x_n)$,

$$\left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array}\right. .$$

we can consider its set of solutions. It is evident that all those solutions are elements of $\mathbb{R}^n$ since the system has n variables. Thus if, for example, we call H said set of solutions, it follows that H is a subset of $\mathbb{R}^n$. The previous system can be written equivalently in matrix form,

$$\left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array}\right. \Leftrightarrow \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, we also have the following matrix representation for H:

$$H \equiv \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The subsets, H, obtained in this way as the set of solutions of a homogeneous linear system of equations constitute what are called vector subspaces. Let's see it more precisely in the following definition.

Definición 22. Consider the set $\mathbb{R}^n$.

We call a vector subspace of $\mathbb{R}^n$ any subset $H \subseteq \mathbb{R}^n$, formed by the solutions of a homogeneous linear system with variables $(x_1, \ldots, x_n)$,

$$\left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array}\right.$$

We will then say that said system constitutes a set of implicit equations for the vector subspace H and we will use the notation

$$H \equiv \left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array}\right.$$

In the previous definition we can write the implicit equations using their matrix expression,

$$AX = 0,$$

where,

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m\times n},\ 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m\times 1},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then in abbreviated form we have that

$$H \equiv AX = 0.$$

A solution of this system is any element $p \in \mathbb{R}^n$ that, substituted into the matrix equation, verifies it, i.e., that satisfies $Ap = 0$ and then H, which is formed by the set of solutions of said system, will be more precisely:

$$H = \{p \in \mathbb{R}^n \text{ such that } Ap = 0\}.$$

A fundamental aspect in the theory of vector subspaces lies in the fact that they can always be obtained as the set of linear combinations of certain vectors. In fact, the basis for making this statement has already been laid in the topic dedicated to solving systems.

Take a vector subspace of $\mathbb{R}^n$,

$$H \equiv AX = 0, \qquad \text{where } A \in \mathcal{M}_{m\times n}, 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The elements of H are the solutions of the system formed by its implicit equations, $AX = 0$ and therefore to determine those elements we have to solve this system. To do this, first we will calculate the number of parameters needed. Suppose the system has a d-dimensional solution and therefore needs d parameters, $\alpha_1, \alpha_2, \ldots, \alpha_d$, to be solved. Applying any of the methods seen in Chapter 5 we will arrive at a general solution of the system, written using the parameters, which put in column has the form

$$\begin{pmatrix} v_{1,1}\alpha_1 + v_{1,2}\alpha_2 + \cdots + v_{1,d}\alpha_d \\ v_{2,1}\alpha_1 + v_{2,2}\alpha_2 + \cdots + v_{2,d}\alpha_d \\ \vdots \\ v_{n,1}\alpha_1 + v_{n,2}\alpha_2 + \cdots + v_{n,d}\alpha_d \end{pmatrix},$$

where $v_{i,j}$ $i = 1, \ldots, n$, $j = 1, \ldots, d$ are certain constants that will appear during the solving process. Now, if we use the properties of the sum and product of tuples by a number we have

$$\begin{pmatrix} v_{1,1}\alpha_1 + v_{1,2}\alpha_2 + \cdots + v_{1,d}\alpha_d \\ v_{2,1}\alpha_1 + v_{2,2}\alpha_2 + \cdots + v_{2,d}\alpha_d \\ \vdots \\ v_{n,1}\alpha_1 + v_{n,2}\alpha_2 + \cdots + v_{n,d}\alpha_d \end{pmatrix} = \alpha_1 \begin{pmatrix} v_{1,1} \\ v_{2,1} \\ \vdots \\ v_{n,1} \end{pmatrix} + \alpha_2 \begin{pmatrix} v_{1,2} \\ v_{2,2} \\ \vdots \\ v_{n,2} \end{pmatrix} + \cdots + \alpha_d \begin{pmatrix} v_{1,d} \\ v_{2,d} \\ \vdots \\ v_{n,d} \end{pmatrix}.$$

If we call

$$v_1 = \begin{pmatrix} v_{1,1} \\ v_{2,1} \\ \vdots \\ v_{n,1} \end{pmatrix}, \quad v_2 = \begin{pmatrix} v_{1,2} \\ v_{2,2} \\ \vdots \\ v_{n,2} \end{pmatrix}, \ldots \quad v_d = \begin{pmatrix} v_{1,d} \\ v_{2,d} \\ \vdots \\ v_{n,d} \end{pmatrix},$$

we obtain

$$\begin{pmatrix} v_{1,1}\alpha_1 + v_{1,2}\alpha_2 + \cdots + v_{1,d}\alpha_d \\ v_{2,1}\alpha_1 + v_{2,2}\alpha_2 + \cdots + v_{2,d}\alpha_d \\ \vdots \\ v_{n,1}\alpha_1 + v_{n,2}\alpha_2 + \cdots + v_{n,d}\alpha_d \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d.$$

That is, all the solutions of the system $AX = 0$ that are obtained by giving values to $\alpha_1, \alpha_2, \ldots, \alpha_d$ in the previous equality are written as a linear combination with coefficients $\alpha_1, \alpha_2, \ldots, \alpha_d$ of the vectors $v_1, v_2, \ldots, v_d$. H is the set of all those solutions and therefore

$$\begin{aligned} H &= \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d : \alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{R}\} \\ &= \langle v_1, v_2, \ldots, v_d \rangle. \end{aligned}$$

In summary, we have demonstrated the following property

Propiedad 23. *For every vector subspace $H \subseteq \mathbb{R}^n$ there exist $v_1, v_2, \ldots, v_d \in \mathbb{R}^n$ such that*

$$H = \langle v_1, v_2, \ldots, v_d \rangle.$$

We then say that the vector subspace H is generated by $v_1, v_2, \ldots, v_d$ or also that $v_1, v_2, \ldots, v_d$ are a generating system of H.

Furthermore, the arguments we used before provide us with a method to calculate the vectors $v_1, v_2, \ldots, v_d$.

Ejemplos 24.
1) Consider the vector subspace of $\mathbb{R}^4$

$$H \equiv \begin{cases} x + y + z + w = 0 \\ x - y + z - 2w = 0 \end{cases}$$

and let's try to represent it through the set of linear combinations of certain tuples. To do this, first we solve the system formed by the implicit equations of H. Using any of the techniques from Chapter 5, we can see that it is a system with a 2-dimensional solution. Using the parameters α and β we can write the solution in the form

$$(\frac{-2\alpha + \beta}{2}, \frac{-3\beta}{2}, \alpha, \beta).$$

Using the properties of the sum and product of matrices, we have that

$$(\frac{-2\alpha + \beta}{2}, \frac{-3\beta}{2}, \alpha, \beta) = \alpha(-1, 0, 1, 0) + \beta(\frac{1}{2}, -\frac{3}{2}, 0, 1)$$

and therefore

$$H = \langle(-1, 0, 1, 0), (\frac{1}{2}, -\frac{3}{2}, 0, 1)\rangle.$$

2) Let's now calculate a generating system for the vector subspace of $\mathbb{R}^5$

$$H \equiv \begin{cases} x_1 + x_2 - 2x_3 + 2x_4 + x_5 = 0 \\ 5x_1 + 3x_2 - 3x_3 + 4x_4 + 2x_5 = 0 \\ 3x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 0 \\ 2x_1 + 4x_2 - 2x_3 + x_4 = 0 \\ 3x_1 + 4x_2 - 2x_3 + x_4 = 0 \end{cases}.$$

Again we begin by solving the system. We can do this using any of the techniques from Chapter 5, obtaining as a result that the system has a 1-dimensional solution. As a function of the parameter α, all solutions of the system are written as

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha).$$

Furthermore,

$$(0, \frac{1}{9}\alpha, -\frac{1}{9}\alpha, -\frac{2}{3}\alpha, \alpha) = \alpha(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1).$$

Therefore,

$$H = \langle(0, \frac{1}{9}, -\frac{1}{9}, -\frac{2}{3}, 1)\rangle.$$

The following property is noteworthy:

Propiedad 25. *$\mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$. Furthermore, it holds that*

$$\mathbb{R}^n = \langle e_1, e_2, \ldots, e_n\rangle,$$

where $e_1, e_2, \ldots, e_n$ are the coordinate n-tuples of $\mathbb{R}^n$,

$$e_1 = (1, 0, 0, \ldots, 0, 0), \quad e_2 = (0, 1, 0, \ldots, 0, 0), \quad \ldots, \quad e_n = (0, 0, 0, \ldots, 0, 1).$$

Proof.
We can see that $\mathbb{R}^n$ is precisely the set of solutions of the homogeneous linear system with n variables $(x_1, x_2, \ldots, x_n)$, of equations

$$\begin{cases} 0x_1 + \cdots + 0x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0x_1 + \cdots + 0x_n = 0 \end{cases}$$

Indeed, since the previous homogeneous linear system holds for any values of $x_1, x_2, \ldots, x_n$, we have that the set of solutions H of said system is given by

$$H = \{(\alpha_1, \alpha_2, \ldots, \alpha_n)\},$$

for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, that is, H is precisely $\mathbb{R}^n$.

Since

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1(1, 0, 0, \ldots, 0, 0) + \alpha_2(0, 1, 0, \ldots, 0, 0) + \cdots \alpha_n(0, 0, 0, \ldots, 0, 1),$$

we have that

$$\mathbb{R}^n = \langle (1, 0, 0, \ldots, 0, 0), (0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1) \rangle,$$

a result we already saw in part **ii)** of **Property 155** in Chapter 4.

□

5.5.2 Basis and dimension

Every vector subspace of $\mathbb{R}^n$ can be written from the set of linear combinations of certain vectors $v_1, v_2, \ldots, v_d \in \mathbb{R}^n$ that generate it in the form

$$H = \langle v_1, v_2, \ldots, v_d \rangle.$$

Of course, it is desirable that this representation be as simple as possible and involve as few vectors as possible. We know that if the vectors $v_1, v_2, \ldots, v_d$ are dependent, we can eliminate some of them and still generate the same set of linear combinations. On the contrary, if they are independent, in $\langle v_1, v_2, \ldots, v_d \rangle$ we cannot eliminate any vector without losing combinations. In other words, the ideal would be to have a representation of H through a set of generating vectors that were independent since that would guarantee that said representation is the simplest possible. This motivates the concept of a basis of a vector subspace.

Definición 26. We call a basis of the vector subspace $H \subseteq \mathbb{R}^n$ any set of vectors $\{v_1, v_2, \ldots, v_d\} \subseteq \mathbb{R}^n$ such that:

- $H = \langle v_1, v_2, \ldots, v_d \rangle$. That is, they are a generating system of H.
- $\{v_1, v_2, \ldots, v_d\}$ is independent.

The number of vectors $v_1, v_2, \ldots, v_d$ that form part of a basis is an invariant of the corresponding vector subspace and does not depend on the technique used to calculate them. That is why we can give the following definition:

Definición 27. We call the dimension of the vector subspace $H \subseteq \mathbb{R}^n$, and denote it $\dim(H)$, the number of vectors that makes up any of its bases.

Ejemplo 28.

Consider the vector subspace of $\mathbb{R}^4$

$$H \equiv \begin{cases} x - 2z + 3w = 0 \\ y - 4z + 6w = 0 \end{cases}$$

Solving the system, it is easy to see that a generating system of H is given by

$$\{(2,4,1,0),(1,2,2,1)\}.$$

It is evident that $(2,4,1,0)$ and $(1,2,2,1)$ are independent. Consequently

$$\{(2,4,1,0),(1,2,2,1)\}$$

is a basis of H and therefore, since two vectors appear in the basis, we have that $\dim(H) = 2$.

Note that:

- Given the subspace through some implicit equations, solving the system using any of the methods from Chapter 5, the procedure we saw on page 212 always provides us with a basis.
- The number of elements in the basis coincides with the number of parameters resulting from solving the system, i.e., if the implicit equations of a vector subspace H are a system with a d-dimensional solution then $\dim(H) = d$.

It is easy to prove the following property that makes possible the calculation of the dimension of a subspace without the need to calculate its basis.

Propiedad 29. *Consider the vector subspace $H \subseteq \mathbb{R}^n$ given by its implicit equations*

$$H \equiv \left\{ \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array} \right. . \textit{ Then:}$$

$$\dim(H) = n - \text{rango} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

Ejemplo 30.

Given the vector subspace of $\mathbb{R}^4$

$$H \equiv \left\{ \begin{array}{l} x - 2z + 3w = 0 \\ y - 4z + 6w = 0 \end{array} \right. ,$$

since

$$\text{rango} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 6 \end{pmatrix} = 2,$$

we have that

$$\dim(H) = 4 - 2 = 2.$$

Nota. In $\mathbb{R}^n$ we can consider, for $i = 1, \dots, n$, the coordinate vectors (the coordinate n-tuples)

$$e_i = (0, \dots, 0, \overset{i)}{1}, 0, \dots, 0) \in \mathbb{R}^n.$$

It is easy to check that $B_c = \{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$ since they are independent and $\mathbb{R}^n = \langle e_1, e_2, \dots, e_n \rangle$. We will call the basis B_c the canonical basis of $\mathbb{R}^n$ and since it has n elements we conclude that

$$\dim(\mathbb{R}^n) = n.$$

The following property allows us to save some calculations when finding a basis of a subspace, knowing its dimension:

Propiedad 31. *Let $H \subseteq \mathbb{R}^n$ be a vector subspace with dimension d. Then:*

1. *Any set of vectors of H with more than d elements is always dependent.*
2. *No set of vectors of H with fewer than d elements can generate all of H.*
3. *A set of vectors with d elements that is independent generates all of H (and is therefore a basis of H).*
4. *A generating system of H with d elements is independent (and therefore a basis of H).*

In view of the previous property and given that, according to what we indicated before, the dimension of $\mathbb{R}^n$ is n, we recall here as a consequence some of the results that appeared in **Property 155** in Chapter 4:

Propiedad 32.

1. *Any set of vectors of $\mathbb{R}^n$ with more than n elements is always dependent.*
2. *No set of vectors of $\mathbb{R}^n$ with fewer than n elements can generate all of $\mathbb{R}^n$.*
3. *A set of n vectors of $\mathbb{R}^n$ that is independent generates all of $\mathbb{R}^n$ (and is therefore a basis of $\mathbb{R}^n$).*
4. *A generating system of $\mathbb{R}^n$ with n elements is independent (and therefore a basis of $\mathbb{R}^n$).*

5.5.3 Coordinates with respect to a basis

Consider a vector subspace $H \subseteq \mathbb{R}^n$ and let $\{v_1, v_2, \ldots, v_d\}$ be a basis of H. In such a case we know that

$$H = \langle v_1, v_2, \ldots, v_d \rangle.$$

Then, given any element $q \in H$ we will have that

$$q = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d$$

for certain coefficients $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{R}$. We may ask if q can admit two different representations of this type. That is, if it will be possible that

$$q = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_d v_d$$

for $\beta_1, \beta_2, \ldots, \beta_d \in \mathbb{R}$ different from the previous values $\alpha_1, \alpha_2, \ldots, \alpha_d$. The point is that if this were true we arrive at

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d = q = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_d v_d$$

$$\Rightarrow (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_d - \beta_d)v_d = 0$$

and, since $v_1, v_2, \ldots, v_d$ are independent, finally

$$\begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_2 \\ \vdots \\ \alpha_d = \beta_d \end{cases}$$

Necessarily the two representations of q must coincide. In other words, the representation of any element of a vector subspace with respect to a given basis is unique. The coefficients that appear in that representation (and which we have seen are unique since once the basis is fixed they only depend on q) are what we call the coordinates of q with respect to the basis.

Definición 33. Given the vector subspace H and the basis $B = \{v_1, v_2, \ldots, v_d\}$ of H, we call the coordinates of $v \in H$ with respect to the basis B the ordered coefficients $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d.$$

Ejemplos 34.
1) Take the canonical basis of $\mathbb{R}^n$, $B_c = \{e_1, e_2, \ldots, e_n\}$. Given any vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ we have that

$$v = (v_1, v_2, \ldots, v_n) = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

and therefore the coordinates of v with respect to B_c are $(v_1, v_2, \ldots, v_n)$. That is, the coordinates of any vector of $\mathbb{R}^n$ with respect to the canonical basis are itself.

2) Consider the vector subspace H that has as a basis

$$B = \{(2,0,1,0), (-1,1,1,1), (0,1,-1,1)\}.$$

Suppose we know that $(5,2,-2,2) \in H$ and that we need to calculate its coordinates with respect to B. We have that the coordinates will be the coefficients (α, β, γ) such that

$$(5,2,-2,2) = \alpha(2,0,1,0) + \beta(-1,1,1,1) + \gamma(0,1,-1,1).$$

Performing the indicated operations in this equality,

$$(5,2,-2,2) = (2\alpha - \beta, \beta + \gamma, \alpha + \beta - \gamma, \beta + \gamma)$$

$$\Leftrightarrow \begin{cases} 2\alpha - \beta = 5 \\ \beta + \gamma = 2 \\ \alpha + \beta - \gamma = -2 \\ \beta + \gamma = 2 \end{cases}$$

Solving this system we obtain that $\alpha = 2$, $\beta = -1$ and $\gamma = 3$. Therefore the coordinates of $(5,2,-2,2)$ are $(2,-1,3)$.

In fact, as we see in the previous example, the calculation of coordinates with respect to a basis always reduces to solving a linear system of equations.