Chapter 3

Integration of Functions

We have seen that the derivative represents the rate of change of a function. Hence, we can later interpret the derivative in different ways, such as the speed of variation of a certain phenomenon that evolves with respect to time. In numerous situations, it is easier to determine the growth rate than the total value reached by a magnitude. In those cases, we must devise mechanisms to, from the rate function, deduce the total value function at each instant. This is where the concept of indefinite and definite integrals comes into play. In this case, the geometric interpretation of the definite integral as the area enclosed by a function will also lead us to different applications of the concept in various contexts.

3.1 Indefinite Integration

Integration is the inverse process of differentiation. Given a function $f(x)$, we can calculate its derivative $f'(x)$. Now what we intend to do is to calculate a function $F(x)$ whose derivative coincides with $f(x)$, that is, $F'(x) = f(x)$. This is what in the following definition we call an antiderivative of $f(x)$.

Definición 1. Given a set $D \subseteq \mathbb{R}$ and a function $f : D \to \mathbb{R}$, we call an primitive or antiderivative of f any differentiable function, $F : D \to \mathbb{R}$, such that

$$F'(x) = f(x), \ \forall x \in D.$$

Ejemplo 2. Given the function $f(x) = x$, we can calculate different antiderivatives:

$$F(x) = \frac{x^2}{2}, \quad F_1(x) = \frac{x^2}{2} + 3, \quad F_2(x) = \frac{x^2}{2} - 10.$$

It is evident that $F'(x) = F_1'(x) = F_2'(x) = x$.

It can be shown that any continuous function has at least one antiderivative. More specifically, we have the following property:

Propiedad 3. *Given $f : (a,b) \to \mathbb{R}$ continuous, there always exists $F : (a,b) \to \mathbb{R}$ differentiable such that*

$$F'(x) = f(x), \quad \forall x \in (a,b).$$

Therefore, the set of functions for which an antiderivative exists is broad and includes all elementary functions, all functions obtained by composition or operation of elementary functions, and piecewise-defined functions that are continuous.

Actually, in **Example 2** we see that the same function has infinitely many antiderivatives, but all of them must follow a common pattern. If $F(x)$ and $G(x)$ are antiderivatives of the same function, $f(x)$, then at every point we will have

$$G'(x) = f(x) = F'(x) \Rightarrow (G - F)'(x) = 0.$$

But if a function has a derivative of zero, it must be a constant, so

$$G(x) - F(x) = C \in \mathbb{R} \Rightarrow G(x) = F(x) + C.$$

Thus, given a fixed antiderivative $F(x)$, all others are obtained by adding a constant, just as happens in the last example. This same idea is reflected more precisely below.

Propiedad 4. *Let $I \subseteq \mathbb{R}$ be an interval and a function $f : I \to \mathbb{R}$, then, if the function $F : I \to \mathbb{R}$ is an antiderivative of f, we have that*

$$G : I \to \mathbb{R} \textit{ is an antiderivative of } f \Leftrightarrow G = F + C,$$

where $C \in \mathbb{R}$.

Therefore, given $f(x)$, the representation of all its antiderivatives will be of the form

$$F(x) + C,$$

where C is a constant that can take any value. This one-parameter representation (a single parameter appears) is what we call the indefinite integral in the next definition.

Definición 5. Given an interval $I \subseteq \mathbb{R}$ and the function $f : I \to \mathbb{R}$, we call the indefinite integral of f and denote it by

$$\int f \, dx \quad \text{or} \quad \int f(x) \, dx$$

any one-parameter formula that allows us, by giving values to its parameter, which we will call the constant of integration, to obtain all the antiderivatives of the function f.

Observe that, given a function $f : I \to \mathbb{R}$, if we know one of its antiderivatives, F, then taking into account the above, the indefinite integral of f will be

$$\int f \, dx = F + C,$$

where C is the constant of integration.

Ejemplo 6. Consider the function $f(x) = x$ defined on all $\mathbb{R}$. It is evident that the function

$$F(x) = \frac{1}{2}x^2$$

is an antiderivative of f. Then, the indefinite integral of f will be

$$\int x \, dx = \frac{1}{2}x^2 + C.$$

If in the previous expression we give real values to the parameter C, we can obtain all the antiderivatives of the function $f(x) = x$.

Given a function $f : I \to \mathbb{R}$ differentiable on the interval I, it is clear that

$$\int f'(x) \, dx = f(x) + C.$$

3.2 Calculation of the Indefinite Integral

Although we have seen in the previous section that every continuous function has an antiderivative and therefore an indefinite integral, the explicit calculation of that antiderivative is not always straightforward. The calculation of the indefinite integral of a function is a complicated problem that requires knowledge, firstly, of the derivatives of all elementary functions and the rules of differentiation and, secondly, of specific methods for the integration of more complicated functions. The knowledge of the properties of differentiation and of the elementary functions allows us to obtain the following integration rules, which are, after all, the direct translation of the properties seen in Chapter 2:

As always, there are two essential points:

1. We must know the integral of the elementary functions.

2. We need properties of the integral for operations and composition.

Regarding the first point, we have only to invert the table of derivatives to obtain a table of integrals for many of the elementary functions:

- $\displaystyle\int x^n \; dx = \frac{1}{n+1}x^{n+1} + C, \; \forall n \in \mathbb{N}.$
- $\displaystyle\int x^\alpha \; dx = \frac{1}{\alpha+1}x^{\alpha+1} + C, \; \forall \alpha \in \mathbb{R} - \{-1\}.$
- $\displaystyle\int e^{kx} \; dx = \frac{1}{k}e^{kx} + C, \; \forall k \in \mathbb{R} - \{0\}.$
- $\displaystyle\int a^{kx} \; dx = \frac{1}{k\log(a)}a^{kx} + C, \; \forall k \in \mathbb{R} - \{0\}, a \in \mathbb{R}.$
- $\displaystyle\int \frac{1}{x} \; dx = \log(x) + C.$
- $\displaystyle\int \cos(x) \; dx = \operatorname{sen}(x) + C.$
- $\displaystyle\int \operatorname{sen}(x) \; dx = -\cos(x) + C.$
- $\displaystyle\int \frac{1}{\cos^2(x)} \; dx = \tan(x) + C.$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \; dx = \operatorname{arcsen}(x) + C, \int \frac{-1}{\sqrt{1-x^2}} \; dx = \arccos(x) + C.$
- $\displaystyle\int \frac{1}{1+x^2} \; dx = \operatorname{arctg}(x) + C.$
- $\displaystyle\int \cosh(x) \; dx = \operatorname{senh}(x) + C.$
- $\displaystyle\int \operatorname{senh}(x) \; dx = \cosh(x) + C.$

The main problem of integration lies in the second point since, contrary to differentiation, we only have properties for the sum and the product by a number.

- Given the functions $f, g : I \to \mathbb{R}$, defined on the interval I, it holds that:
  - $\displaystyle\int (f+g)dx = \int f\ dx + \int g\ dx.$
  - $\displaystyle\int k \cdot f\ dx = k \cdot \int f\ dx.$

This lack of properties makes the integration problem considerably more difficult than that of differentiation.

It is evident that most functions fall outside points 1. and 2. above. We must then devise mechanisms that allow reducing the integral of any function to the integral of the functions we have just seen. For this, we will fundamentally use two methods:

- integration by substitution,
- integration by parts.

Integration by Change of variable

Consider the indefinite integral of the function f,

$$\int f(x)\ dx.$$

This integral is expressed in terms of the variable x. However, it could be interesting to express it in terms of another variable that is related to x by a certain formula. Suppose that the variable t is related to the variable x by the equation

$$\varphi(t) = \phi(x).$$

Let's differentiate this expression using the following mnemonic rule in which we introduce the differential with respect to t, dt, and the differential with respect to x, dx,

$$\varphi'(t)dt = \phi'(x)dx.$$

If we combine these two equalities we obtain two equations,

$$\begin{cases} \varphi(t) = \phi(x), \\ \varphi'(t)\ dt = \phi'(x)\ dx, \end{cases}$$

through which we can solve for x in terms of t and dx in terms of dt and t to subsequently substitute the results obtained into the integral we intend to calculate. The integral is solved in terms of the variable t and then the substitution is reversed.

Actually, the substitution method is justified by the following theorem, which would be a direct translation of the chain rule for differentiation.

Teorema 7.

Given $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ such that $f(I) \subseteq J$, we have that

$$\int g'(f(x)) \cdot f'(x)\ dx = (g \circ f)(x).$$

Ejemplos 8.

1) Calculate $\displaystyle\int (2x+1)e^{x^2+x}\ dx$.

$$\begin{aligned}\int (2x+1)e^{x^2+x}\ dx &= \left(\begin{array}{l} t = x^2+x \\ dt = (2x+1)dx \Rightarrow dx = \frac{dt}{2x+1}\end{array}\right) = \int e^t\ dt = e^t + C \\ &= \left(\begin{array}{c}\text{reversing the}\\ \text{substitution}\end{array}\right) = e^{x^2+x} + C.\end{aligned}$$

2) Calculate $\displaystyle\int \frac{1}{x\sqrt{1-\mathrm{Ln}^2(x)}}\ dx$.

$$\begin{aligned}&\int \frac{1}{x\sqrt{1-\mathrm{Ln}^2(x)}}\ dx = \\ &= \left(\begin{array}{lll}\mathrm{sen}(t) = \mathrm{Ln}(x) & \Rightarrow & t = \mathrm{arcsen}\big(\mathrm{Ln}(x)\big) \\ \cos(t)\ dt = \frac{1}{x}\ dx & \Rightarrow & dx = x\cos(t)\ dt\end{array}\right) = \int \frac{x\cos(t)\ dt}{x\sqrt{1-\mathrm{sen}^2(x)}} \\ &= \int \frac{\cos(t)\ dt}{\sqrt{\cos^2(x)}} = \int \frac{\cos(t)dt}{\cos(t)} = \int dt = t + C = \left(\begin{array}{c}\text{reversing the}\\ \text{substitution}\end{array}\right) = \mathrm{arcsen}(\mathrm{Ln}(x)) + C.\end{aligned}$$

Integration by Parts

The method of integration by parts is based on the properties of differentiation of the product of functions. We know that

$$\int (f(x)\cdot g(x))'\ dx = f(x)\cdot g(x) + C$$

and also

$$\int (f(x)\cdot g(x))'\ dx = \int (f'(x)\cdot g(x) + f(x)\cdot g'(x))\ dx = \int f'(x)\cdot g(x)\ dx + \int f(x)\cdot g'(x)\ dx$$

so that combining the two equalities

$$\int f'(x)\cdot g(x)\ dx + \int f(x)\cdot g'(x)\ dx = f(x)\cdot g(x) + C$$

from which

$$\int f(x)\cdot g'(x)\ dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\ dx + C$$

and finally, including the integration parameter, C, in the indefinite integral of the second member, we have demonstrated the following:

Propiedad 9. *Let $f, g : I \to \mathbb{R}$ be functions differentiable on the interval I, then*

$$\int f(x)\cdot g'(x)\ dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\ dx.$$

The previous property is not used directly but through the following scheme, which is called the **method of integration by parts** for the calculation of the integral $\int f(x)g(x)dx$:

$$\int (\underbrace{f(x)}_{=u(x)}\cdot\underbrace{g(x)}_{=v'(x)})\ dx = \int (u(x)\cdot v'(x))\ dx = \left\{\begin{array}{lll} u(x) = f(x) & \Rightarrow & u'(x) = f'(x) \\ v'(x) = g(x) & & v(x) = \int g(x)\ dx\end{array}\right\}$$

$$= \begin{pmatrix} \text{using the} \\ \text{property} \end{pmatrix} = u(x) \cdot v(x) - \int u'(x) \cdot v(x)\ dx.$$

Ejemplo 10.

$$\begin{aligned} \int x \log(x)\ dx &= \left\{ \begin{array}{lcl} u = \log(x) & & u' = (\log(x))' = \frac{1}{x} \\ & \Rightarrow & \\ v' = x & & v = \int x\ dx = \frac{1}{2}x^2 \end{array} \right\} = \\ &= \frac{1}{2}x^2 \log(x) - \int \frac{1}{x}\frac{x^2}{2}\ dx = \frac{x^2}{2}\log(x) - \frac{1}{2}\int x\ dx = \frac{x^2}{2}\log(x) - \frac{x^2}{4} + C. \end{aligned}$$

The method of integration by parts can be applied to obtain the integral of functions of the type

$$p(x) \cdot f(x),$$

where $p(x)$ is a polynomial and $f(x)$ is a logarithmic, exponential, or trigonometric function. In such cases, it may be necessary to apply integration by parts successively to obtain the result. The use of the integration by parts method is also indicated for the calculation of the integral of the product of a trigonometric function by an exponential.

Ejemplos 11.

1) Calculate $\int (x^2 + x - 1)e^x\ dx$.

$$\begin{aligned} &\int (x^2 + x - 1)e^x\ dx = \\ &\quad = \left\{ \begin{array}{l} u = x^2 + x - 1 \\ v' = e^x \end{array} \Rightarrow \begin{array}{l} u' = 2x + 1 \\ v = \int e^x\ dx = e^x \end{array} \right\} = (x^2 + x - 1)e^x \\ &\quad - \int (2x + 1)e^x\ dx = \left\{ \begin{array}{l} u = 2x + 1 \\ v' = e^x \end{array} \Rightarrow \begin{array}{l} u' = 2 \\ v = \int e^x\ dx = e^x \end{array} \right\} \\ &\quad = (x^2 + x - 1)e^x - \left((2x + 1)e^x - \int 2e^x\ dx \right) \\ &\quad = (x^2 + x - 1)e^x - (2x + 1)e^x + 2e^x + C = (x^2 - x)e^x + C. \end{aligned}$$

2) Calculate $\int \log(x)\ dx$.

Note that

$$\log(x) = 1 \cdot \log(x)$$

and we have the product of a polynomial of degree 0 ($p(x) = 1$) by a logarithmic function ($f(x) = \log(x)$), so we will apply integration by parts as follows:

$$\begin{aligned} &\int \log(x)\ dx = \\ &\quad = \left\{ \begin{array}{lcl} u = \log(x) & & u' = \frac{1}{x} \\ & \Rightarrow & \\ v' = 1 & & v = \int 1\ dx = x \end{array} \right\} = x\log(x) - \int \frac{1}{x} x\ dx \\ &\quad = x\log(x) - x + C. \end{aligned}$$

3) Calculate $\int x\cos(x)dx$. We have the product of a polynomial, $p(x) = x$, by a trigonometric function, $f(x) = \cos(x)$. We will solve by integrating by parts:

$$\int x\cos(x)dx = \left\{ \begin{array}{lcl} u = x & & u' = 1 \\ & \Rightarrow & \\ v' = \cos(x) & & v = \int \cos(x)dx = \mathrm{sen}(x) \end{array} \right\}$$

$$= x\mathrm{sen}(x) - \int \mathrm{sen}(x)dx = x\mathrm{sen}(x) + \cos(x) + C.$$

4) We can also use the integration by parts method to calculate the integral of the product of an exponential function by a trigonometric one. In this case, it will be necessary to apply integration by parts twice to be able to solve for the desired integral. Let's see an example.

$$\begin{aligned} &\int e^x\cos(x)dx = \\ &= \left\{ \begin{array}{lcl} u = \cos(x) & & u' = -\mathrm{sen}(x) \\ & \Rightarrow & \\ v' = e^x & & v = \int e^x dx = e^x \end{array} \right\} = e^x\cos(x) + \int e^x\mathrm{sen}(x)dx \\ &= \left\{ \begin{array}{lcl} u = \mathrm{sen}(x) & & u' = \cos(x) \\ & \Rightarrow & \\ v' = e^x & & v = \int e^x dx = e^x \end{array} \right\} \\ &= e^x\cos(x) + e^x\mathrm{sen}(x) - \int e^x\cos(x)dx \end{aligned}$$

In short, we have that

$$\underline{\int e^x\cos(x)dx} = e^x\cos(x) + e^x\mathrm{sen}(x) - \underline{\int e^x\cos(x)dx}$$

and we observe that in both members of the equality we have obtained the integral we intended to calculate. Then, it will be sufficient to solve for it to obtain

$$\int e^x\cos(x)dx = \frac{1}{2}\left(e^x\cos(x) + e^x\mathrm{sen}(x)\right) + C.$$

We will now see a list of techniques that allow us to solve certain specific types of integrals using the two methods we have just seen, substitution and integration by parts, at different points.

3.2.1 Immediate Integrals

When, for solving an integral, we can directly apply some integration rule from the table we saw on page 82, we say that such an integral is an immediate integral. Below we list some types of immediate integrals of interest:

Immediate Integrals of Power Type

These are integrals that can easily be transformed into the form

$$\int f(x)^\alpha f'(x)\ dx = \frac{1}{\alpha+1}f(x)^{\alpha+1} + C.$$

Examples 77.

1) Calculate $\int x^2(3x^3+14)^3$. We have that

$$\int x^2(3x^3+14)^3 = \frac{1}{9}\int 9x^2(3x^3+14)^3\ dx = \left(\begin{array}{l} f(x)=3x^3+14 \\ f'(x)=9x^2 \end{array}\right) = \frac{1}{9}\cdot\frac{1}{4}\cdot(3x^3+14)^4+C.$$

2) Calculate $\int \frac{x-1}{\big((x-1)^2+4\big)^2}\ dx$. In this case we will proceed as follows

$$\begin{aligned}\int \frac{x-1}{\big((x-1)^2+4\big)^2}\ dx &= \frac{1}{2}\int 2(x-1)\big((x-1)^2+4\big)^{-2}\ dx = \left(\begin{array}{l} f(x)=(x-1)^2+4 \\ f'(x)=2(x-1) \end{array}\right) \\ &= \frac{1}{2}\frac{1}{-1}\big((x-1)^2+4\big)^{-1} = \frac{-1}{2\big((x-1)^2+4\big)}+C.\end{aligned}$$

3) Calculate $\int \cos(x)\cdot \operatorname{sen}^3(x)\ dx$. We have that

$$\int \cos(x)\cdot\operatorname{sen}^3(x)\ dx = \left(\begin{array}{l} f(x)=\operatorname{sen}(x) \\ f'(x)=cos(x) \end{array}\right) = \frac{1}{4}\operatorname{sen}^4(x)+C.$$

4) Calculate $\int \frac{x-1}{\big((x-1)^2+4\big)^2}\ dx$.

$$\begin{aligned}\int \frac{x-1}{\big((x-1)^2+4\big)^2}\ dx &= \frac{1}{2}\int 2(x-1)\big((x-1)^2+4\big)^{-2}\ dx = \left(\begin{array}{l} f(x)=(x-1)^2+4 \\ f'(x)=2(x-1) \end{array}\right) \\ &= \frac{1}{2}\frac{1}{-1}\big((x-1)^2+4\big)^{-1} = \frac{-1}{2\big((x-1)^2+4\big)}+C.\end{aligned}$$

5) An integral of the type $\int \frac{1}{(Ax+B)^n}dx$ with $n>1$ can be solved as follows:

$$\begin{aligned}\int \frac{1}{(Ax+B)^n}dx &= \frac{1}{A}\int A(Ax+B)^{-n}dx = \left(\begin{array}{l} f(x)=Ax+B \\ f'(x)=A \end{array}\right) \\ &= \frac{1}{A}\frac{(Ax+B)^{-n+1}}{-n+1} = \frac{1}{A(-n+1)}\frac{1}{(Ax+B)^{n-1}}+C.\end{aligned}$$

For example,

$$\int \frac{1}{(9x-1)^7}dx = \frac{1}{9\cdot(-6)}\frac{1}{(9x-1)^6} = \frac{-1}{54(9x-1)^6}+C.$$

When $n=1$ it is not possible to solve the integral using this method and one must resort to the last section of this chapter where immediate integrals of logarithmic type are treated.

6) Calculate the integral $\int \frac{x}{(Ax+B)^n}dx$ with $n>1$. We have:

$$\int \frac{x}{(Ax+B)^n}dx = \frac{1}{A}\int \frac{Ax+B-B}{(Ax+B)^n}dx$$

$$= \frac{1}{A}\int \frac{1}{(Ax+B)^{n-1}}dx - \frac{B}{A}\int \frac{1}{(Ax+B)^{n}}dx.$$

Now, the integral $\int \frac{1}{(Ax+B)^n}dx$ can be solved following the previous example and we can do the same for $\int \frac{1}{(Ax+B)^{n-1}}$ as long as $n-1>1$. When $n-1=1$ (that is, when $n=2$), as mentioned before, we will have to solve it as indicated in the section dedicated to logarithmic integrals.

Immediate Integrals of Exponential Type

These are integrals that can be adjusted to the form

$$\int f'(x)a^{f(x)}\ dx = \frac{1}{\log(a)}a^{f(x)} + C$$

Examples 78.

1) For $\int \operatorname{sen}(x)e^{\cos(x)}\ dx$. We solve as follows:

$$\int \operatorname{sen}(x)e^{\cos(x)}\ dx = -\int -\operatorname{sen}(x)e^{\cos(x)}\ dx = \begin{pmatrix} f(x)=\cos(x) \\ f'(x) = -\operatorname{sen}(x) \end{pmatrix} = e^{\cos(x)} + C.$$

2) Calculate $\int x^2 \cdot 7^{x^3+5}\ dx$.

$$\int x^2 \cdot 7^{x^3+5}\ dx = \frac{1}{3}\int 3x^2 \cdot 7^{x^3+5}\ dx = \begin{pmatrix} f(x)=x^3+5 \\ f'(x) = 3x^2 \end{pmatrix} = \frac{1}{3\log(7)}7^{x^3+5} + C.$$

Immediate Integrals of Logarithmic Type

These are integrals in which a function appears dividing its derivative. They are of the type

$$\int \frac{f'(x)}{f(x)}\ dx = \log(f(x)) + C.$$

Examples 79.

1) Calculate $\int \tan(x)\ dx$.

$$\int \tan(x)\ dx = -\int \frac{-\operatorname{sen}(x)}{\cos(x)}\ dx = \begin{pmatrix} f(x)=\cos(x) \\ f'(x) = -\operatorname{sen}(x) \end{pmatrix} = -\log(\cos(x)) + C.$$

2) Obtain $\int \frac{4x^3+6x^2+1}{x^4+2x^3+x-1}\ dx$. We have that

$$\int \frac{4x^3+6x^2+1}{x^4+2x^3+x-1}\ dx = \begin{pmatrix} f(x)=x^4+2x^3+x-1 \\ f'(x) = 4x^3+6x^2+1 \end{pmatrix} = \log(x^4+2x^3+x-1) + C.$$

3) Let's calculate the integral $\int \frac{1}{Ax+B}dx$ for certain constants$A, B \in \mathbb{R}$.

$$\int \frac{1}{Ax+B}\ dx = \frac{1}{A}\int \frac{A}{Ax+B}\ dx = \left(\begin{array}{l} f(x) = Ax+B \\ f'(x) = A \end{array} \right) = \frac{1}{A}\log(Ax+B) + C.$$

For example,

$$\int \frac{1}{3x+6}dx = \frac{1}{3}\log(3x+6) + C.$$

4) Calculate $\int \frac{x}{Ax+B}dx$.

$$\int \frac{x}{Ax+B}dx = \frac{1}{A}\int \frac{Ax+B-B}{Ax+B}dx = \frac{1}{A}\int \frac{Ax+B}{Ax+B}dx - \frac{B}{A}\int \frac{1}{Ax+B}dx$$
$$= \frac{x}{A} - \frac{B}{A^2}\log(Ax+B).$$

3.2.2 Integration of Rational Functions (only real roots in the denominator)

A rational function is a function of the type $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are two polynomials.

To calculate $\int \frac{p(x)}{q(x)}dx$, the idea is to express the rational function $\frac{p(x)}{q(x)}$ as a sum of simple fractions. We do this by following the steps indicated below in two different cases:

a) Calculation of $\int \frac{\mathbf{p(x)}}{\mathbf{q(x)}}\mathbf{dx}$ **when grado(p(x))** $\geq$ **grado(q(x)).**

If grado$(p(x)) \geq$ grado$(q(x))$ we perform the division of $p(x)$ by $q(x)$ so that we obtain a remainder $r(x)$ of degree lower than that of $q(x)$. Schematically we have:

$$\begin{array}{cc} p(x) & \underline{\vert\, q(x)} \\ r(x) & s(x) \end{array} \Rightarrow \left\{ \begin{array}{l} \dfrac{p(x)}{q(x)} = s(x) + \dfrac{r(x)}{q(x)} \\ \\ \text{grado}(r(x)) < \text{grado}(q(x)) \end{array} \right. .$$

Subsequently we will perform the integral of the obtained expression,

$$\int \left(s(x) + \frac{r(x)}{q(x)} \right)\ dx = \int s(x)dx + \int \frac{r(x)}{q(x)}dx.$$

We see that the integral of the polynomial $s(x)$ appears, which is easy to calculate. Also appears the integral of the rational function $\frac{r(x)}{q(x)}$ but now we have that the degree of the numerator (degree of $r(x)$) is less than the degree of the denominator (degree of $q(x)$). To solve this last integral we proceed as indicated in case **b)**.

Example 80. Calculate the integral

$$\int \frac{x^4 - 2x^3 - 7x^2 + 21x - 12}{x^3 - 5x^2 + 8x - 4}dx.$$

This is the integral of a rational function. Since the degree of the numerator is greater than that of the denominator we will divide both polynomials:

$$\begin{array}{cc} x^4 - 2x^3 - 7x^2 + 21x - 12 & \underline{\vert\, x^3 - 5x^2 + 8x - 4} \\ x & x+3 \end{array} .$$

That is,

$$\begin{aligned}\frac{x^4-2x^3-7x^2+21x-12}{x^3-5x^2+8x-4} &= \\ &= x+3+\frac{x}{x^3-5x^2+8x-4}\end{aligned}$$

and therefore

$$\begin{aligned}\int \frac{x^4-2x^3-7x^2+21x-12}{x^3-5x^2+8x-4} &= \\ &= \int (x+3)dx + \int \frac{x}{x^3-5x^2+8x-4}dx \\ &= \frac{x^2}{2}+3x+\int \frac{x}{x^3-5x^2+8x-4}dx.\end{aligned}$$

The last integral remains to be solved, in which a rational function appears but now with the degree of the numerator lower than the degree of the denominator.

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b) Calculation of $\int \frac{\mathbf{p(x)}}{\mathbf{q(x)}}\mathbf{dx}$ when grado$(\mathbf{p(x)}) <$ grado$(\mathbf{q(x)})$.

If grado$(p(x)) <$ grado$(q(x))$ we will decompose the rational function into a sum of simple fractions in the form

$$f(x)=\frac{p(x)}{q(x)}=S_1+S_2+\cdots+S_k.$$

To determine which are the fractions S_1, S_2,... S_k we follow the following steps:

1. We will calculate all the solutions of the polynomial equation

$$q(x)=0.$$

Ejemplo 12. Continuing with **Example 80**, to solve the pending integral, we will set the denominator to zero

$$x^3-5x^2+8x-4=0$$

and we will calculate the solutions of the equation thus obtained. The calculation of these solutions is done using Ruffini's method as follows:

$$\begin{array}{c|cccc} & 1 & -5 & 8 & -4 \\ 1 & & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \underline{|0} \\ 2 & & 2 & -4 & \\ \hline & 1 & -2 & \underline{|0} & \\ 2 & & 2 & & \\ \hline & 1 & \underline{|0} & & \end{array}$$

We obtain the solution $x=2$ twice (that is $x=2$ with multiplicity 2) and the solution $x=1$ once ($x=1$ with multiplicity 1).

---

2. For each real solution, $\alpha \in \mathbb{R}$ with multiplicity $k \in \mathbb{N}$, we will add to the decomposition of the rational function the following group of simple fractions

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_k}{(x-\alpha)^k}.$$

Note that if there is multiplicity of k we will add k simple fractions for the solution α.

The coefficients $A_1, A_2, \ldots, A_k$ are, in principle, unknown and will have to be calculated once we know all the simple fractions involved in the decomposition of the function we are integrating.

Ejemplo 13.
1) Continuing with example 12, let's see which simple fractions we will add for each solution:

- The solution $\alpha = 2$ has multiplicity 2. For it we will add two simple fractions,
$$\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2}.$$
- The solution $\alpha = 1$ has multiplicity 1. For it we add only one simple fraction,
$$\frac{A_3}{x-1}.$$

We have studied the fractions to add for each solution. We will gather all of them to obtain the decomposition into simple fractions of the function we were trying to integrate:

$$\frac{x}{x^3-5x^2+8x-4} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{x-1}. \tag{3.1}$$

The coefficients A_1, A_2 and A_3 must now be calculated. Taking into account that

$$\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{x-1} =$$
$$\frac{A_1(x-2)(x-1) + A_2(x-1) + A_3(x-2)^2}{(x-2)^2(x-1)}$$

and that
$$x^3-5x^2+8x-4 = (x-2)^2(x-1),$$
equation (3.1) leads us to

$$x = A_1(x-2)(x-1) + A_2(x-1) + A_3(x-2)^2.$$

By giving different values to the variable x in the previous equality we obtain a system of equations from which we will solve for said coefficients:

$$\left.\begin{array}{ll} x=1 \Rightarrow & 1 = A_3 \\ x=2 \Rightarrow & 2 = A_2 \\ x=0 \Rightarrow & 0 = 2A_1 - A_2 + 4A_3 \end{array}\right\}$$

Solving this system we obtain the following solutions:

$$A_1 = -1,\ A_2 = 2,\ A_3 = 1.$$

Therefore, substituting these values, the decomposition into simple fractions is

$$\frac{x}{x^3-5x^2+8x-4} =$$
$$= \frac{-1}{x-2} + \frac{2}{(x-2)^2} + \frac{1}{x-1}.$$

The integral of all the simple fractions that appeared can be performed using methods indicated above. Thus, the first and the third are immediate of logarithmic type and the second is immediate of power type. Thus,

$$\int \frac{x}{x^3-5x^2+8x-4} =$$

$$
\begin{aligned}
&= -\int \frac{1}{x-2} + 2\int \frac{1}{(x-2)^2} + \int \frac{1}{x-1} \\
&= -\log(x-2) - 2(x-2)^{-1} + \log(x-1) + C.
\end{aligned}
$$

3.3 Definite Integral

The indefinite integral of a function $f(x)$ is, up to an integration constant, an antiderivative of the function $F(x)$. If we take an interval (a, b) on which the function $f(x)$ is defined, we will define in this section what is called the definite integral of $f(x)$ between a and b. While the indefinite integral is a function (the antiderivative function), the definite integral will be a number.

Due to its important applications, the definite integral is a fundamental tool in mathematics and other disciplines. The relationship between the definite integral and the indefinite integral is established in some of the most important results of mathematical analysis such as the Fundamental Theorem of Calculus, Barrow's rule, etc. The study of these results is beyond the objectives of this course but we will rely on them to give a simple definition of the definite integral.

Let's see below the precise definition of the definite integral.

Definición 14.

i) Let $f : D \to \mathbb{R}$ be a real function and let $a, b \in \mathbb{R}$, $a < b$, such that $(a, b) \subseteq D$. Suppose that f is continuous and bounded on (a, b) and that there exists a function $F : [a, b] \to \mathbb{R}$ continuous such that it is differentiable on (a, b) and that $\forall x \in (a, b)$
$$F'(x) = f(x),$$
that is, F is an antiderivative of f on (a, b). Then we call the definite integral of f between a and b the real number given by
$$\int_a^b f(x)\ dx = F(b) - F(a).$$

ii) Let $f : D \to \mathbb{R}$ be a real function, let $a, b \in \mathbb{R}$, $a < b$, such that $(a, b) \subseteq D$ and suppose that f is bounded on (a, b) and continuous on (a, b) except at most at the points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ then we call the definite integral of f between a and b the real number given by
$$\int_a^b f(x)\ dx = \int_{x_0}^{x_1} f(x)\ dx + \int_{x_1}^{x_2} f(x)\ dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\ dx.$$
We define the definite integral of f between b and a as
$$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx.$$

Given $f : D \to \mathbb{R}$ bounded and given $a \in \mathbb{R}$ we define the definite integral of f between a and a as
$$\int_a^a f(x)\ dx = 0.$$

The difference $F(b) - F(a)$ is usually denoted as $[F(x)]_a^b$ so we have $\displaystyle\int_a^b f(x)dx = [F(x)]_a^b$.

With this definition, the classical properties of the definite integral arise immediately. In particular, the Fundamental Theorem of Calculus will be a consequence of the definition and of **Property ??** from Chapter 2 for the differentiation of piecewise defined functions. Likewise, the substitution formula is a direct consequence of the chain rule for the differentiation of functions.

Propiedades 15.

i) Let $f, g : D \to \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$, such that $(a, b) \subseteq D$, so that f and g are in the conditions of part ii) of **Definition 14** *for said interval, then:*

1. $\forall \alpha, \beta \in \mathbb{R}$, $\displaystyle\int_a^b (\alpha f + \beta g)\ dx = \alpha \int_a^b f(x)\ dx + \beta \int_a^b f(x)\ dx.$
2. $\forall c \in [a, b]$, $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$
3. *(Fundamental Theorem of Calculus) If we consider the function*

$$\begin{array}{c} F : [a, b] \to \mathbb{R} \\ F(x) = \displaystyle\int_a^x f(t)\ dt \end{array},$$

then F is a continuous function on $[a, b]$ and differentiable at all those points $x \in (a, b)$ where f is continuous, having in such cases that

$$F'(x) = f(x).$$

4. *If we modify the function f on a finite set of points, the value of its integral between a and b does not change.*

ii) Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b) then

$$\int_a^b f'(x)\ dx = f(b) - f(a).$$

iii) (Substitution formula) Let $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to [a, b]$ in the conditions of part ii) of **Definition 14** *for their respective domains. Suppose that g is differentiable on $[c, d]$, then*

$$\int_{g(c)}^{g(d)} f(x)dx = \int_c^d f(g(x))g'(x)dx.$$

3.3.1 Applications of the Definite Integral

Calculation of Area from the Definite Integral

The definite integral between two points allows calculating the area enclosed by the function over the interval determined by them.

Propiedad 16. *Given the function $f : (a, b) \to \mathbb{R}$ in the conditions of part ii) of* **Definition 14***, the area between the axis $y = 0$, the vertical lines $x = a$ and $x = b$ and the graph of the function f is calculated by the definite integral*

$$\int_a^b f(x)dx$$

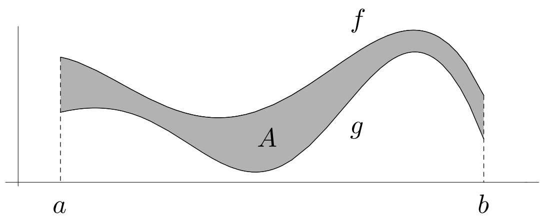
so that if f is positive on (a, b) said integral will provide the area with a positive sign and if f is negative it will do so with a negative sign.

As indicated in the property, since $\int_a^b -f(x)dx = -\int_a^b f(x)dx$, if the function f is negative, the definite integral between a and b will provide the value of the area but with a negative sign. This must be taken into account when working with functions that change sign.

Given two functions $f, g : [a, b] \to$ such that

$$f(x) \geq g(x),\ \forall x \in [a, b],$$

we can use the last property to calculate the area, A, between the graphs of both functions.



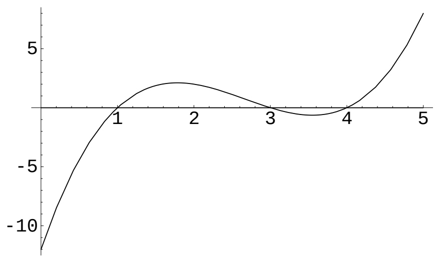
It is clear that A will be the difference between the area under f and the area under g. Therefore,

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$

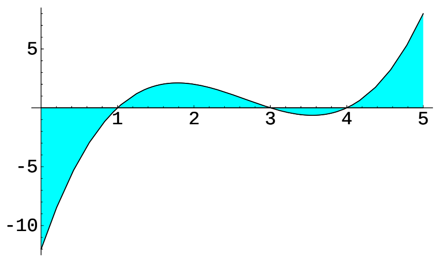
In this regard, it is again necessary to take into account the possible intersection points between the functions f and g that could change the sign of the previous integral.

Examples 81.

1) Let's calculate the area enclosed between the function $f(x) = x^3 - 8x^2 + 19x - 12$ and the x-axis over the interval $[0, 5]$. If we represent the function $f(x)$ in the indicated interval we obtain the graph



The area enclosed by the function will therefore be the region that appears shaded in the following figure:



We know that the area enclosed by a function in an interval is calculated by performing the definite integral of the function in that interval. However, we observe in both graphs that the function $f(x)$ presents several sign changes in the interval $[0, 5]$ so we will not be able to calculate the area directly by performing the integral

$$\int_0^5 f(x)dx.$$

We must first determine in which intervals the function is positive or negative. Although in this case it is possible to observe with the naked eye in the graphical representation of $f(x)$ the intervals in which it is positive or negative, we will not always have the graph of the function available, so we will proceed as if we did not have it.

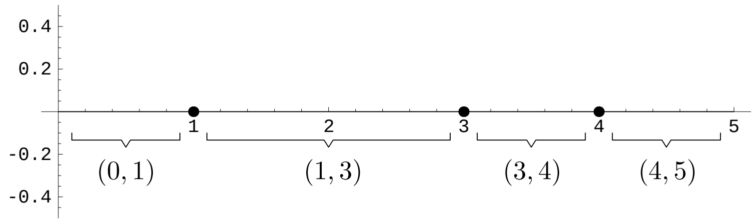
Thus, we must determine when

$$f(x) > 0 \quad \text{and} \quad f(x) < 0.$$

To do this, we begin by solving the equation $f(x) = 0$, that is,

$$x^3 - 8x^2 + 19x - 12 = 0.$$

If we apply Ruffini's method, it is easy to verify that the solutions of this equation are $x = 1$, $x = 3$ and $x = 4$ which divide the interval $[0, 5]$ into four subintervals



and we know, as a consequence of Bolzano's Theorem (see page 38), that within each of these intervals the function $f(x)$ cannot change sign. It is then enough to check the sign of the function at one point in each interval to deduce that

$$\begin{cases} f(x) < 0 \text{ in } (0,1). \\ f(x) > 0 \text{ in } (1,3). \\ f(x) < 0 \text{ in } (3,4). \\ f(x) > 0 \text{ in } (4,5). \end{cases}$$

Therefore, in the intervals $(0, 1)$ and $(3, 4)$ the definite integral will provide the area enclosed by the function f but with a negative sign. To calculate the area correctly we must change the sign of the result of the definite integral over these two intervals. In this way we will obtain the exact value of the area enclosed by $f(x)$ over the interval $[0, 5]$ as follows

$$\text{área} = -\int_0^1 f(x)dx + \int_1^3 f(x)dx - \int_3^4 f(x)dx + \int_4^5 f(x)dx.$$

Since the indefinite integral of $f(x)$ is

$$\int f(x)dx = \int (x^3 - 8x^2 + 19x - 12)dx = \frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x + C,$$

finally we have

$$\text{área} = -\left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x\right]_0^1 + \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x\right]_1^3 - \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x\right]_3^4$$

$$+ \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{19}{2}x^2 - 12x\right]_4^5 = \frac{133}{12} = \frac{59}{12} + \frac{8}{3} + \frac{5}{12} + \frac{37}{12} = 11.0833.$$

2) Let's calculate the area between the functions $f_1(x) = x^2 - 2x + 2$ and $f_2(x) = -x^2 + 4x - 1$ over the interval $[1, 3]$. We know that the area between both functions is obtained by the definite integral

$$\int_1^3 (f_1(x) - f_2(x))dx.$$

However, again we must take into account the possible sign changes that will be determined by the intersections between the two functions. In this case we must determine if

$$f_1(x) - f_2(x) < 0 \quad \text{or} \quad f_1(x) - f_2(x) > 0$$

and to do this we begin by solving the equation $f_1(x) - f_2(x) = 0$, that is,

$$x^2 - 2x + 2 - (-x^2 + 4x - 1) = 0 \Leftrightarrow 2x^2 - 6x + 3 = 0.$$

Applying the formula for the quadratic equation we verify that the solutions of this equation are

$$\begin{cases} x = \frac{3-\sqrt{3}}{2} = 0.6339 \\ x = \frac{3+\sqrt{3}}{2} = 2.3660 \end{cases}.$$

Of these two solutions, only the second is in the interval $[1, 3]$ that interests us, dividing it into two subintervals, $(1, \frac{3+\sqrt{3}}{2})$ and $(\frac{3+\sqrt{3}}{2}, 3)$. Again it is sufficient to check one point from each of these intervals to deduce that

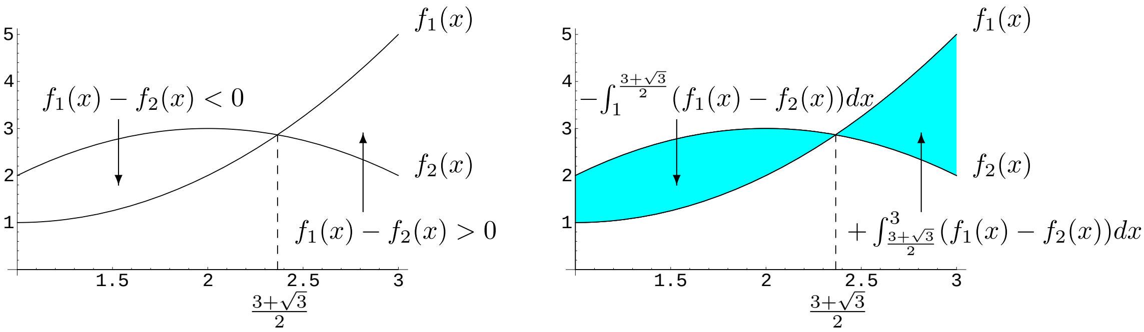
$$\begin{cases} f_1(x) - f_2(x) < 0 \text{ in } (1, \frac{3+\sqrt{3}}{2}). \\ f_1(x) - f_2(x) > 0 \text{ in } (\frac{3+\sqrt{3}}{2}, 3). \end{cases}$$

So we will obtain the desired area by compensating the negative sign of the definite integral in the first interval as follows:

$$\text{área} = -\int_1^{\frac{3+\sqrt{3}}{2}} (f_1(x) - f_2(x))dx + \int_{\frac{3+\sqrt{3}}{2}}^3 (f_1(x) - f_2(x))dx$$

$$= -\left[\frac{2}{3}x^3 - 3x^2 + 3x\right]_1^{\frac{3+\sqrt{3}}{2}} + \left[\frac{2}{3}x^3 - 3x^2 + 3x\right]_{\frac{3+\sqrt{3}}{2}}^3 = \frac{2}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{2}{3} + \sqrt{3} = 2.3987.$$

If we observe the graph of both functions in the interval $[1, 3]$ and the region they delimit, we can verify that they indeed intersect at the point $\frac{3+\sqrt{3}}{2}$ of the interval $[1, 3]$ so it is necessary to change the sign of the definite integral in the first subinterval as appears in the graph.



Calculation of the Total Value Function from the Velocity Function

We already mentioned at the beginning of this Chapter that in many occasions one has the function that determines the velocity of a certain phenomenon but not the total or accumulated value function. In such circumstances, part *ii)* of **Property 15** allows recovering the accumulated value function if we have some initial data.

Suppose that a certain phenomenon that evolves over time is determined by a magnitude, $M(t)$, for which we know its rate of change, $v(t)$, at each instant. Suppose also that we know that at time t_0 said magnitude took a value M_0. We try to determine what the function $M(t)$ is from the following information:

$$\begin{cases} M'(t) = v(t), \\ M(t_0) = M_0. \end{cases}$$

Using part *ii)* of **Definition 15** we have that

$$\int_{t_0}^{t} M'(t)dt = M(t) - M(t_0) \Rightarrow M(t) - M_0 = \int_{t_0}^{t} v(t)dt \Rightarrow M(t) = M_0 + \int_{t_0}^{t} v(t)dt.$$

This last identity provides us with the data, initially unknown, of the value $M(t)$ at any instant.

Example 82. A study is carried out on the garbage generated in a certain city during the first month of the year. Suppose that the function $B : [0, 30] \to \mathbb{R}$ given by

$$B(t) = \frac{t^3}{200} - \frac{t^2}{4} + 3t + 30$$

provides the amount of garbage produced on day t, measured in tons. We then have that $B(t)$ expresses in tons/day the velocity of garbage production we had on day t. Since the function $B(t)$ provides different values for different days, the velocity of garbage production would have changed from one day to the next. Suppose that initially ($t_0 = 0$) the amount of garbage accumulated in the public landfill was 3200 tons. We now intend to calculate the function $M(t)$ that provides the total amount of tons accumulated in the landfill up to day t. Based on the previous comments we have that

$$\begin{aligned} M(t) &= M_0 + \int_{t_0}^{t} B(x)dx = 3200 + \int_0^t \left(\frac{x^3}{200} - \frac{x^2}{4} + 3x + 30\right) dx = 3200 + \left[\frac{x^4}{800} - \frac{x^3}{12} + \frac{3x^2}{2} + 30x\right]_0^t \\ &= 3200 + \left(\frac{t^4}{800} - \frac{t^3}{12} + \frac{3t^2}{2} + 30t - 0\right) = \frac{t^4}{800} - \frac{t^3}{12} + \frac{3t^2}{2} + 30t + 3200. \end{aligned}$$

For example, on day $t = 15$ the amount of garbage accumulated in the landfill will be equal to

$$M(15) = 3769.53$$

while on day $t = 30$ we have

$$M(30) = 4212.5.$$

Calculation of the Average Value of a Function

Given a function $f : [a, b] \to \mathbb{R}$, initially positive, we know that the integral

$$\int_a^b f(x)dx$$

is the area enclosed by the function over the interval $[a, b]$. We can ask ourselves if it is possible to find a constant function $g(x) = k$ that encloses in the same interval the same area as the function f. We have that the area for g is

$$\int_a^b g(x)dx = \int_a^b kdx = [kt]_a^b = k(b - a).$$

If we want it to enclose the same area as f it must hold that

$$k(b - a) = \int_a^b f(x)dx \Rightarrow k = \frac{1}{b - a}\int_a^b f(x)dx.$$

Therefore the constant function $g(x) = \frac{1}{b-a}\int_a^b f(x)dx$, encloses the same area as $f(x)$. Said constant is what is usually called the average value of the function f.

Definition 83. Given a function $f : [a, b] \to \mathbb{R}$, we call the average value of the function the quantity

$$\frac{1}{b - a}\int_a^b f(x)dx.$$

Examples 84.
1) In **Example 82** we saw how the function $B(t)$ provided the velocity of garbage production in tons/day for day t. The average velocity during the first 30 days (during the first month of the year over which the study was conducted) will be

$$\text{Average velocity} = \text{Average value of } B(t) = \frac{1}{30-0}\int_0^{30} B(t)dt = \frac{1}{30}\int_0^{30}\left(\frac{t^3}{200} - \frac{t^2}{4} + 3t + 30\right) dt$$
$$= \frac{1}{30}\left[\frac{t^4}{800} - \frac{t^3}{12} + \frac{3t^2}{2} + 30t\right]_0^{30} = 33.75.$$

2) During the 365 days of a year, the balance of a certain bank account is given by the function

$$L : [0, 365] \to \mathbb{R}$$
$$L(t) = 1000 + 1000\cos\left(\tfrac{2\pi}{365}t\right)$$

so that $L(t)$ are the euros that were in the account on day t. The average balance of said account will be the average value of the function $L(t)$ in the interval $[0, 365]$ which we calculate as follows:

$$\text{Average of } L = \frac{1}{365-0}\int_0^{365} L(t)dt = \frac{1000}{365}\int_0^{365}\left(1 + \cos\left(\frac{2\pi}{365}t\right)\right) dt$$
$$= \frac{1000}{365}\left[t + \frac{365}{2\pi}\,\text{sen}\left(\frac{2\pi}{365}t\right)\right]_0^{365} = 1000.$$

Therefore, the average balance of the account is 1000 euros.

Continuously Compounded Variable Interest

There are numerous situations in which the interest earned by a certain capital varies over time. In fact, mortgages, investment funds, etc. with variable interest are all products offered by any financial institution.

Let us now assume that we have a certain initial capital P_0 in an account that pays interest that is not fixed. The interest, I, will vary over time so that in reality what we will have is a function $I(t)$ that provides the interest (we will assume in decimal form) in each year t. We then intend to calculate the formula for the function $P(t)$ that yields the accumulated capital up to year t.

Studying the relationship that exists between the capital function $P(t)$ and the interest function $I(t)$, it can be shown that

$$\boxed{P(t) = P_0 e^{\int_{t_0}^{t} I(t)dt}}\,, \tag{3.2}$$

which is the formula for capital subject to continuously compounded, variable interest, $I(t)$, (in decimal form) when the capital in the initial year t_0 is P_0.

The interest we study in this section is compound since the capitals received as interests in turn produce new interests for the following years. Furthermore, it is continuously compounded since the payment of interests occurs continuously. In the case of variable interest paid in a determined number of periods (instead of continuously) it is not possible to obtain simple formulas like this one we have achieved here.

Examples 85.
1) We invest 1000€ in an investment fund that offers variable interest compounded continuously. The initial interest of the fund is 3% per year and each year it increases by 0.5%. Calculate the capital we will receive after 5 years.

Evidently, in this case the interest is variable since initially we have 0.03 per unit of interest that increases by 0.005 annually. It is evident that to calculate the interest in year t we can apply the formula

$$\underbrace{0.03}_{\text{Initial interest}} + \underbrace{0.005}_{\text{Annual increment}} \times \underbrace{t}_{\text{Years elapsed}} .$$

Therefore,

$$I(t) = 0.03 + 0.005t.$$

Since we do not know in which specific year the investment was made, to simplify we will accept that it was made in the year $t_0 = 0$. Then, if we apply formula (3.2), the capital will be given by

$$P(t) = 1000 \cdot e^{\int_0^t (0.03+0.005t)dt} = 1000 \cdot e^{0.03t+0.005\frac{t^2}{2}}.$$

Therefore, after 5 years the capital will be

$$P(5) = 1000 \cdot e^{0.03\cdot 5+0.005\frac{5^2}{2}} = 1236.77€$$

2) A certain bank account pays variable interest compounded continuously. The accumulated capital in that account is given by the function

$$P(t) = 1500 \cdot e^{0.01(t+1)^2}.$$

What interest have we received each year?. Specifically, what interest will we obtain in the third year?.

In this case we know the capital function $P(t)$ and we must calculate the interest function $I(t)$. To do this we will use formula (3.3):

$$I(t) = \frac{P'(t)}{P(t)} = \frac{\left(1500 \cdot e^{0.01(t+1)^2}\right)'}{1500 \cdot e^{0.01(t+1)^2}} = \frac{1500 \cdot 0.01 \cdot 2(t+1) \cdot e^{0.01(t+1)^2}}{1500 \cdot e^{0.01(t+1)^2}} = 0.02(t+1).$$

Thus, the interest function is $I(t) = 0.02(t+1)$ and using it we calculate the interest in year three,

$$I(3) = 0.02(3+1) = 0.08,$$

or what is the same, 8%.

3.4 Material Adicional

3.4.1 Integración de funciones racionales (caso general)

Ampliación de conceptos sobre integración de funciones racionales. Página 88

Como ya dijimos en la subsección 3.2.2, una función racional es una función del tipo $\frac{p(x)}{q(x)}$, donde $p(x)$ y $q(x)$ son dos polinomios.

Comenzamos viendo varios casos de funciones racionales que pueden ser integradas de forma sencilla:

1) Integrales de la forma

$$\int \frac{1}{(Ax+B)^n}dx, \quad \int \frac{x}{(Ax+B)^n}dx, \quad n \in \mathbb{N}.$$

Ya hemos visto que estas integrales pueden resolverse como integrales inmediatas de tipo potencial o logarítmico.

2) Integral del tipo $\displaystyle\int \frac{1}{(x-a)^2+b^2}\,dx$. Resolveremos haciendo un cambio de variable para transformala en la integral inmediata $\int \frac{1}{1+x^2}dx = \arctan(x)+C$ (está en la tabla de la página 81). Veámoslo:

$$\begin{aligned}
&\int \frac{1}{(x-a)^2+b^2}\,dx = \\
&\quad = \int \frac{1}{b^2\left(\frac{(x-a)^2}{b^2}+1\right)}\,dx = \frac{1}{b^2}\int \frac{1}{\left(\frac{x-a}{b}\right)^2+1}\,dx \\
&\quad = \left(\begin{array}{l} t=\frac{x-a}{b} \\ dt=\frac{1}{b}\,dx \Rightarrow dx = b\,dt \end{array} \right) = \frac{1}{b^2}\int \frac{1}{t^2+1} b\,dt = \frac{1}{b}\int \frac{1}{t^2+1}\,dt \\
&\quad = \frac{1}{b}\operatorname{arctg}(t) = \left(\begin{array}{c} \text{deshaciendo el} \\ \text{cambio} \end{array} \right) = \frac{1}{b}\operatorname{arctg}\left(\frac{x-a}{b}\right).
\end{aligned}$$

3) Integral del tipo $\displaystyle\int \frac{x}{(x-a)^2+b^2}dx$. Puede resolverse utilizando el apartado anterior y los métodos para integrales inmediatas de tipo logarítmico. Para ello hacemos lo siguiente:

$$\begin{aligned}
&\int \frac{x}{(x-a)^2+b^2}dx = \\
&\quad = \int \frac{x-a+a}{(x-a)^2+b^2}dx = \int \frac{x-a}{(x-a)^2+b^2}dx + a\int \frac{1}{(x-a)^2+b^2}dx.
\end{aligned}$$

Ahora bien, la integral $\int \frac{1}{(x-a)^2+b^2}dx$ puede calcularse como en el apartado **2)** y la integral $\int \frac{x-a}{(x-a)^2+b^2}dx$ es de tipo logarítmico ya que,

$$\int \frac{x-a}{(x-a)^2+b^2}dx = \left(\begin{array}{l} f(x)=(x-a)^2+b^2 \\ f'(x)=2(x-a) \end{array} \right) = \frac{1}{2}\log\left((x-a)^2+b^2\right)+C.$$

A partir de estas integrales sencillas podemos intentar resolver otras integrales racionales más complejas. Análogamente a como hicimos en la subsección 3.2.2, para calcular $\int \frac{p(x)}{q(x)}dx$, la idea es expresar la función racional $\frac{p(x)}{q(x)}$ como la suma de fracciones simples de las que aparecen en los apartados **1)**, **2)** y **3)** que acabamos de ver. Ello lo hacemos siguiendo los pasos que indicamos a continuación en dos casos distintos:

a) Cálculo de $\displaystyle\int \frac{\mathbf{p(x)}}{\mathbf{q(x)}}\mathbf{dx}$ cuando grado(p(x)) $\geq$ grado(q(x)).

Si grado$(p(x)) \geq$ grado$(q(x))$ efectuamos la división de $p(x)$ entre $q(x)$ de manera que obtengamos un resto $r(x)$ de grado inferior al de $q(x)$. Esquemáticamente tenemos:

$$\begin{array}{cc} p(x) & \underline{|\,q(x)} \\ r(x) & s(x) \end{array} \Rightarrow \left\{ \begin{array}{l} \dfrac{p(x)}{q(x)} = s(x) + \dfrac{r(x)}{q(x)} \\ \\ \text{grado}(r(x)) < \text{grado}(q(x)) \end{array} \right. .$$

Posteriormente efectuaremos la integral de la expresión obtenida,

$$\int \left(s(x)+\frac{r(x)}{q(x)}\right)\,dx = \int s(x)dx + \int \frac{r(x)}{q(x)}dx.$$

Vemos que aparece la integral del polinomio $s(x)$ que es fácil de calcular. También aparece la integral de la función racional $\frac{r(x)}{q(x)}$ pero ahora tenemos que el grado del numerador (grado de $r(x)$) es menor que el grado del denominador (grado de $q(x)$). Para resolver esta última integral procedemos como se indica en el caso **b)**.

Ejemplo 17. Calcular la integral

$$\int \frac{x^6 - 9x^5 + 30x^4 - 33x^3 - 25x^2 + 69x - 28}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} dx.$$

Se trata de la integral de una función racional. Puesto que el grado del numerador es mayor que el del denominador dividiremos ambos polinomios:

$$\begin{array}{r|l} x^6 - 9x^5 + 30x^4 - 33x^3 - 25x^2 + 69x - 28 & x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20 \\ \hline x^4 - 15x^2 + 27x - 8 & x + 1 \end{array}.$$

Es decir,

$$\begin{aligned} &\frac{x^6 - 9x^5 + 30x^4 - 33x^3 - 25x^2 + 69x - 28}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} = \\ &= x + 1 + \frac{x^4 - 15x^2 + 27x - 8}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} \end{aligned}$$

y por tanto

$$\begin{aligned} &\int \frac{x^6 - 9x^5 + 30x^4 - 33x^3 - 25x^2 + 69x - 28}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} = \\ &= \int (x+1)dx + \int \frac{x^4 - 15x^2 + 27x - 8}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} dx \\ &= \frac{x^2}{2} + x + \int \frac{x^4 - 15x^2 + 27x - 8}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} dx. \end{aligned}$$

Queda pendiente de resolver la última integral en la que aparece una función racional paro ahora con grado del numerador inferior al grado del denominador.

b) Cálculo de $\int \frac{\mathbf{p(x)}}{\mathbf{q(x)}} \mathbf{dx}$ cuando grado$(\mathbf{p(x)}) <$ grado$(\mathbf{q(x)})$.

Si grado$(p(x)) <$ grado$(q(x))$ descompondremos la función racional en una suma de fracciones simples en la forma

$$f(x) = \frac{p(x)}{q(x)} = S_1 + S_2 + \cdots + S_k,$$

donde las expresiones S_1, S_2,... S_k son del tipo indicado en los apartados **1)**, **2)** y **3)** de esta subsección. Para determinar cuáles son las fracciones S_1, S_2,... S_k seguimos los siguientes pasos:

1. Calcularemos todas las soluciones reales y complejas de la ecuación polinómica

$$q(x) = 0.$$

Ejemplo 18. Siguiendo con el **Ejemplo 17**, para resolver la integral que quedó pendiente, igualaremos a cero el denominador

$$x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20 = 0$$

y calcularemos las soluciones de la ecuación así obtenida. El cálculo de estas soluciones lo hacemos empleando el método de Ruffini como sigue:

	1	-10	39	-72	62	-20
1		1	-9	40	-42	20
	1	-9	30	-42	20	0
1		1	-8	22	-20	
	1	-8	22	-20	0	
2		2	-12	20		
	1	-6	10	0		

Obtenemos la solución $x = 1$ dos veces (es decir $x = 1$ con multiplicidad 2) y la solución $x = 2$ una vez ($x = 2$ con multiplicidad 1), quedando sin resolver el tramo de la ecuación que corresponde al polinomio $x^2 - 6x + 10$. Por tanto, para encontrar todas las soluciones, debemos resolver por último,

$$x^2 - 6x + 10 = 0.$$

Aquí podemos aplicar directamente la fórmula para encontrar las soluciones de una ecuación de segundo grado y obtendremos las soluciones complejas $x = 3 \pm i$ con multiplicidad 1 (es decir $x = 3+i$ y $x = 3-i$ ambas aparecen una sola vez). En definitiva hemos obtenido las siguientes soluciones:

$$\begin{cases} x = 1, & \text{con multiplicidad } 2, \\ x = 2, & \text{con multiplicidad } 1, \\ x = 3 \pm i, & \text{con multiplicidad } 1. \end{cases}$$

---

2. Por cada solución real, $\alpha \in \mathbb{R}$ con multiplicidad $k \in \mathbb{N}$, añadiremos a la descomposición de la función racional el siguiente grupo de fracciones simples

$$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_k}{(x-\alpha)^k}.$$

   Véase que si hay multiplicidad de k añadiremos k fracciones simples para la solución α.

   Los coeficientes $A_1, A_2, \ldots, A_k$ son, en principio, desconocidos y deberán ser calculados una vez sepamos todas las fracciones simples que intervienen en la descomposición de la función que estamos integrando.

3. Por cada par se soluciones complejas, $a \pm bi$ con multiplicidad $k \in \mathbb{N}$, añadiremos a la descomposición los sumandos

$$\frac{M_1 + N_1 x}{(x-a)^2 + b^2} + \frac{M_2 + N_2 x}{\big((x-a)^2 + b^2\big)^2} + \cdots + \frac{M_k + N_k x}{\big((x-a)^2 + b^2\big)^k},$$

   otra vez, tantos sumandos como indique la multiplicidad de la solución.

   Nuevamente los coeficientes $M_1, N_1, \ldots, M_k, N_k$ son desconocidos y se calculan después de haber añadido las fracciones simples correspondientes a todas las soluciones.

Ejemplos 19.
1) Continuando con el ejemplo 18, veamos qué fracciones simples añadiremos para cada solución:

- La solución $\alpha = 1$ tiene multiplicidad 2. Para ella añadiremos dos fracciones simples,

$$\frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}.$$

- La solución $\alpha = 2$ tiene multiplicidad 1. Para ella añadimos una sola fracción simple,

$$\frac{A_3}{x-2}.$$

- La pareja de soluciones complejas $\alpha = 3 \pm i$ tiene multiplicidad 1. Todo número complejo es de la forma $a + bi$. En este caso $a = 3$ y $b = 1$. Añadiremos una fracción del tipo $\frac{Mx+N}{(x-a)^2+b^2}$, es decir,

$$\frac{Mx + N}{(x-3)^2 + 1}.$$

Hemos estudiado las fracciones a añadir para cada solución. Reuniremos todas ella para obtener la descomposición en fracciones simples de la función que intentábamos integrar:

$$\frac{x^4 - 15x^2 + 27x - 8}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} =$$

$$= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x-2} + \frac{Mx+N}{(x-3)^2+1}.$$

Los coeficientes A_1, A_2, A_3, N y M han de ser calculados ahora. Para ello daremos distintos valores a la variable x para obtener un sistema de ecuaciones del que despejaremos dichos coeficientes.

$$\left.\begin{array}{ll} x=0 \Rightarrow & -A_1 + A_2 - \frac{A_3}{2} + \frac{N}{10} = \frac{2}{5} \\ x=-1 \Rightarrow & -\frac{A_1}{2} + \frac{A_2}{4} - \frac{A_3}{3} + \frac{N-M}{17} = \frac{49}{204} \\ x=-2 \Rightarrow & -\frac{A_1}{3} + \frac{A_2}{9} - \frac{A_3}{4} + \frac{N-2M}{26} = \frac{53}{468} \\ x=3 \Rightarrow & \frac{A_1}{2} + \frac{A_2}{4} + A_3 + 3M + N = \frac{19}{4} \\ x=-3 \Rightarrow & -\frac{A_1}{4} + \frac{A_2}{16} - \frac{A_3}{5} + \frac{N-3M}{37} = \frac{143}{2960} \end{array}\right\}$$

Resolviendo este sistema obtenemos las siguientes soluciones:

$$A_1 = -2,\ A_2 = -1,\ A_3 = 1,\ M = 2,\ N = -1.$$

Por tanto, sustituyendo estos valores, la descomposición en fracciones simples es

$$\frac{x^4 - 15x^2 + 27x - 8}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} =$$
$$= \frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2} + \frac{2x-1}{(x-3)^2+1}.$$

La integral de todas la fracciones simples que aparecieron puede realizase empleando métodos indicados anteriormente. Así, la primera y la tercera son inmediatas de tipo logarítmico, la segunda es inmediata de tipo potencial y la última se ajusta a los casos **2)** y **3)** que vimos al principio de esta subsección. Así pues,

$$\begin{aligned} &\int \frac{x^4 - 15x^2 + 27x - 8}{x^5 - 10x^4 + 39x^3 - 72x^2 + 62x - 20} = \\ = \quad & -2\int \frac{1}{x-1} - \int \frac{1}{(x-1)^2} + \int \frac{1}{x-2} + 2\int \frac{x}{(x-3)^2+1} \\ & -\int \frac{1}{(x-3)^2+1}. \\ = \quad & -2\log(x-1) + (x-1)^{-1} + \log(x-2) + 5\arctan(x-3) \\ & + \log\left((x-3)^2+1\right) + C. \end{aligned}$$

2) Calcular la integral

$$\int \frac{x^3 - x^2 + 29}{x^2 - 4x + 9}\ dx.$$

El grado del numerador es mayor que el del denominador así que efectuaremos la división de uno entre el otro

$$\begin{array}{c|c} x^3 - x^2 + 29 & x^2 - 4x + 9 \\ \cline{2-2} 3x + 2 & x + 3 \end{array}$$

de manera que

$$\int \frac{x^3 - x^2 + 29}{x^2 - 4x + 9}\ dx =$$
$$= \int \left(x + 3 + \frac{3x+2}{x^2 - 4x + 9} \right)\ dx = \frac{x^2}{2} + 3x + \int \frac{3x+2}{x^2 - 4x + 9}\ dx.$$

Para calcular esta última integral obtendremos las raíces de la ecuación

$$x^2 - 4x + 9 = 0$$

que al ser una ecuación de segundo grado se resuelve fácilmente siendo sus soluciones $x = 2 + \sqrt{5}i$ y $x = 2 - \sqrt{5}i$ ambas complejas y de multiplicidad 1. Por lo tanto la función racional a integrar se descompondrá en la forma

$$\frac{3x+2}{x^2-4x+9} = \frac{Mx+N}{(x-2)^2+5} = \left(\begin{array}{c} M=3 \\ N=2 \end{array}\right) = \frac{3x+2}{(x-2)^2+5}$$

de manera que

$$\begin{aligned}
&\int \frac{3x+2}{x^2-4x+9}\,dx = \\
&\quad = \int \frac{3x+2}{(x-2)^2+5}\,dx = \int \frac{3(x-2)+8}{(x-2)^2+5}\,dx \\
&\quad = 3\int \frac{x-2}{(x-2)^2+5}\,dx + \int \frac{8}{(x-2)^2+5}\,dx \\
&\quad = \frac{3}{2}\log\left((x-2)^2+5\right) + \frac{8}{\sqrt{5}}\operatorname{arctg}\left(\frac{x-2}{\sqrt{5}}\right) + C.
\end{aligned}$$

Finalmente tendremos que

$$\int \frac{x^3-x^2+29}{x^2-4x+9}\,dx = \frac{x^2}{2} + 3x + \frac{8}{\sqrt{5}}\operatorname{arctg}\left(\frac{x-2}{\sqrt{5}}\right) + \frac{3}{2}\log\left((x-2)^2+5\right) + C.$$

3.4.2 Integración por cambio hasta una integral racional

Ampliación de técnicas de integración. Página 91

Denotaremos mediante $\mathbf{R}(a)$ y $\mathbf{R}(a,b)$ a cualquier expresión que se obtiene mediante suma producto y división de las expresiones a y/ó b y de constantes.

Ejemplos 20.
1) La expresión

$$\frac{\log^3(x) + 2\log^2(x) + 1}{\log^3(x) - 1}$$

es una expresión de tipo $\mathbf{R}(\log(x))$ ya que se obtiene mediante operaciones de suma, producto y división en las que intervienen $\log(x)$ y valores constantes.

2) La expresión

$$\frac{\cos^2(x) + 2x\cos(x) + x}{1 + x^2 + 2\cos^3(x)}$$

es una expresión de tipo $\mathbf{R}(x,\cos(x))$ ya que se obtiene sumando multiplicando o dividiendo las expresiones x, $\cos(x)$ y valores constantes. Sin embargo no es una expresión de tipo $\mathbf{R}(x)$ ni de tipo $\mathbf{R}(\cos(x))$.

Para resolver la integral

$$\int \mathbf{R}(h(x))h'(x)\ dx,$$

donde $h(x)$ puede ser alguna de entre

$$\begin{cases} a^x,\ a>0 \\ e^x \\ \log(x) \\ \arccos(x) \\ \operatorname{arcsen}(x) \\ \operatorname{arctg}(x) \end{cases}$$

efectuaremos el cambio

$$t = h(x).$$

Nota. Una integral del tipo $\int \mathbf{R}(e^x)\ dx$ también se puede resolver utilizando el método anterior.

Ejemplos 21.

1) Calcular $\displaystyle\int \frac{e^x+e^{2x}}{e^x-1}\ dx.$

Se trata de la integral de una expresión del tipo $\mathbf{R}(e^x)$ y por lo tanto tomaremos el cambio $t = e^x$ en el siguiente modo:

$$\begin{aligned}
&\int \frac{e^x+e^{2x}}{e^x-1}\ dx = \\
&\quad = \int \frac{e^x+(e^x)^2}{e^x-1}\ dx = \left(\begin{array}{c} t = e^x \Rightarrow x = \log(t) \\ dx = \frac{1}{t}dt \end{array} \right) = \int \frac{t+t^2}{t-1}\frac{1}{t}dt \\
&\quad = \int \frac{1+t}{t-1}\ dt = \int \left(1+\frac{2}{t-1}\right)\ dt = t + 2\log(t-1) + C \\
&\quad = \left(\begin{array}{c} \text{deshaciendo el} \\ \text{cambio} \end{array} \right) = e^x + 2\log(e^x-1) + C.
\end{aligned}$$

2) Obténgase $\displaystyle\int \frac{1-2\operatorname{arcsen}(x)-\operatorname{arcsen}^2(x)}{\sqrt{1-x^2}\big(1+\operatorname{arcsen}^2(x)\big)^2}\ dx$

En la integral anterior aparece una función del tipo $\mathbf{R}(\operatorname{arcsen}(x))$ por lo que haremos el cambio $t = \operatorname{arcsen}(x)$:

$$\begin{aligned}
&\int \frac{1-2\operatorname{arcsen}(x)-\operatorname{arcsen}^2(x)}{\sqrt{1-x^2}\big(1+\operatorname{arcsen}^2(x)\big)^2}\ dx = \\
&\quad = \left(\begin{array}{l} t = \operatorname{arcsen}(x) \\ dt = \frac{1}{\sqrt{1-x^2}}\ dx \Rightarrow dx = \sqrt{1-x^2}\ dt \end{array} \right) = \int \frac{1-2t-t^2}{\sqrt{1-x^2}(1+t^2)^2}\sqrt{1-x^2}\ dt \\
&\quad = \int \frac{1-2t-t^2}{(1+t^2)^2}\ dt = \int \left(-\frac{2(t-1)}{(1+t^2)^2} - \frac{1}{1+t^2} \right)\ dt \\
&\quad = \int \frac{-2t}{(1+t^2)^2}\ dt + 2\int \frac{1}{(1+t^2)^2}\ dt - \int \frac{1}{1+t^2}\ dt \\
&\quad = (t^2+1)^{-1} + 2\left(\frac{t}{2(1+t^2)} + \frac{1}{2}\operatorname{arctg}(t) \right) - \operatorname{arctg}(t) + C = \frac{t+1}{t^2+1} + C \\
&\quad = \left(\begin{array}{c} \text{deshaciendo el} \\ \text{cambio} \end{array} \right) = \frac{\operatorname{arcsen}(x)+1}{1+\operatorname{arcsen}^2(x)} + C.
\end{aligned}$$

3.4.3 Integración de funciones trigonométricas

Ampliación de técnicas de integración. Página 91

Estudiaremos en esta sección el cálculo de integrales del tipo

$$\int \mathbf{R}(\cos(x), \operatorname{sen}(x))\, dx,$$

donde la expresión **R** tiene el significado que se indica en la página 103. Para resolver esta integral tenemos tres opciones posibles en función de las propiedades que tenga la expresión **R**:

i) Se dice que la expresión **R** es impar en $\operatorname{sen}(x)$ si se verifica que

$$\mathbf{R}(\cos(x), -\operatorname{sen}(x)) = -\mathbf{R}(\cos(x), \operatorname{sen}(x)),$$

lo cual se produce generalmente cuando en **R** aparecen potencias impares de $\operatorname{sen}(x)$. Entonces utilizaremos el cambio:

$$\boxed{\left\{\begin{array}{l} \cos(x) = t \Rightarrow x = \arccos(t) \\ dx = \dfrac{-1}{\sqrt{1-t^2}} dt \end{array}\right.}$$

y además tendremos que

$$\cos^2(x) + \operatorname{sen}^2(x) = 1 \Rightarrow \operatorname{sen}(x) = \sqrt{1-\cos^2(x)} \Rightarrow \boxed{\operatorname{sen}(x) = \sqrt{1-t^2}}.$$

Ejemplo 22. Calcúlese $\displaystyle\int \operatorname{sen}^3(x)\cos^4(x)\, dx$.

Tenemos la integral de una expresión del tipo

$$\mathbf{R}(\operatorname{sen}(x), \cos(x))$$

que es impar en $\operatorname{sen}(x)$ (las potencias de $\operatorname{sen}(x)$ que aparecen son impares). Aplicando el cambio anterior conseguimos:

$$\begin{aligned}
&\int \operatorname{sen}^3(x)\cos^4(x)\, dx = \\
&\quad = (\cos(x) = t) = \int \left(\sqrt{1-t^2}\right)^3 t^4 \frac{-dt}{\sqrt{1-t^2}} = -\int \left(\sqrt{1-t^2}\right)^2 t^4\, dt \\
&\quad = -\int (1-t^2)t^4\, dt = -\int (t^4 - t^6)\, dt = \frac{t^7}{7} - \frac{t^5}{5} + C \\
&\quad = \left(\begin{array}{c} \text{deshaciendo el} \\ \text{cambio} \end{array}\right) = \frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + C.
\end{aligned}$$

---

ii) Se dice que la expresión **R** es impar en $\cos(x)$ si se verifica que

$$\mathbf{R}(-\cos(x), \operatorname{sen}(x)) = -\mathbf{R}(\cos(x), \operatorname{sen}(x))$$

lo cual se produce generalmente cuando en **R** aparecen potencias impares de $\cos(x)$. Entonces utilizaremos el cambio:

$$\boxed{\left\{\begin{array}{l} \operatorname{sen}(x) = t \Rightarrow x = \operatorname{arcsen}(t) \\ dx = \dfrac{1}{\sqrt{1-t^2}} dt \end{array}\right.}$$

y además tendremos que

$$\cos^2(x) + \operatorname{sen}^2(x) = 1 \Rightarrow \cos(x) = \sqrt{1-\operatorname{sen}^2(x)} \Rightarrow \boxed{\cos(x) = \sqrt{1-t^2}}.$$

Ejemplo 23. Resolver la integral $\displaystyle\int \frac{1}{\cos(x)}\, dx$.

Tenemos que

$$\frac{1}{\cos(x)}$$

es una expresión del tipo $\mathbf{R}(\cos(x), \mathrm{sen}(x))$ impar en $\cos(x)$ (las potencias de $\cos(x)$ que aparecen son impares) y por lo tanto utilizaremos el cambio $\mathrm{sen}(x) = t$:

$$\begin{aligned}
&\int \frac{1}{\cos(x)}\, dx = \\
&\quad = (\mathrm{sen}(x) = t) = \int \frac{1}{\sqrt{1-t^2}} \frac{dt}{\sqrt{1-t^2}} = \int \frac{1}{1-t^2}\, dt \\
&\quad = \int \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right) dt = \frac{1}{2}\log(t+1) - \frac{1}{2}\log(t-1) + C \\
&\quad = \begin{pmatrix} \text{deshaciendo el} \\ \text{cambio} \end{pmatrix} = \frac{1}{2}\log\left(\frac{\mathrm{sen}(x)+1}{\mathrm{sen}(x)-1}\right) + C.
\end{aligned}$$

iii) Independientemente de que la expresión $\mathbf{R}$ sea par o impar en $\mathrm{sen}(x)$ ó en $\cos(x)$ siempre podremos aplicar el siguiente cambio que se denomina usualmente **cambio general**

$$\boxed{\left\{\begin{array}{l} t = \tan\left(\dfrac{x}{2}\right) \Rightarrow x = 2\mathrm{arctg}(t) \\[2ex] dx = \dfrac{2dt}{1+t^2} \end{array}\right.}$$

Utilizando las fórmulas trigonométricas habituales, de las ecuaciones del cambio se deducen las siguientes expresiones para $\cos(x)$ y $\mathrm{sen}(x)$:

$$\left\{\begin{array}{l} \boxed{\cos(x) = \dfrac{1-t^2}{1+t^2}} \\[3ex] \boxed{\mathrm{sen}(x) = \dfrac{2t}{1+t^2}} \end{array}\right.$$

Ejemplo 24. Calcular $\displaystyle\int \frac{1}{1+\cos(x)+2\mathrm{sen}(x)}\, dx$.

$$\begin{aligned}
&\int \frac{1}{1+\cos(x)+2\mathrm{sen}(x)}\, dx \\
&\quad = \left(t = \tan\left(\frac{x}{2}\right)\right) = \int \frac{1}{1+\frac{1-t^2}{1+t^2}+2\frac{2t}{1+t^2}} \frac{2dt}{1+t^2} = \int \frac{1+t^2}{1+t^2+1-t^2+4t} \frac{2dt}{1+t^2} \\
&\quad = \int \frac{2dt}{2+4t} = \int \frac{dt}{1+2t} = \frac{1}{2}\log(1+2t) + C \\
&\quad = \begin{pmatrix} \text{deshaciendo el} \\ \text{cambio} \end{pmatrix} = \frac{1}{2}\log\left(1+2\tan\left(\frac{x}{2}\right)\right) + C.
\end{aligned}$$

3.4.4 El área como integral definida

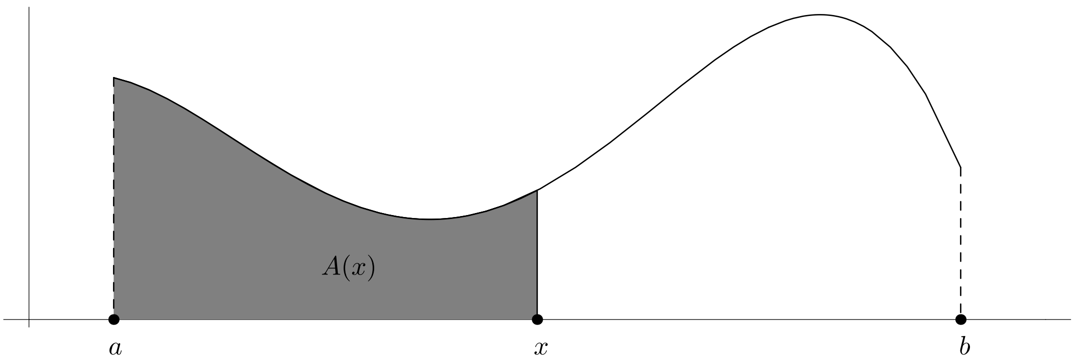
Ampliación del cálculo del área a partir de la integral definida. Página 92

En el apartado 3.3.1 expusimos como, a partir de la integral definida, podemos calcular el área de una región plana. Nos disponemos ahora a demostrar aquel resultado.

La definición del concepto de área ha constituido un problema de envergadura a lo largo de la historia de las matemáticas. Nosotros no profundizaremos en el aspecto matemático de esa cuestión y nos contentaremos con admitir que existen ciertos subconjuntos, $A \subseteq \mathbb{R}^2$, a los cuales se les puede asociar un número, área(A), que llamaremos área verificando las siguientes propiedades:

- El área del recinto encerrado por un rectángulo cuyos lados miden l_1 y l_2 es el producto $l_1 \cdot l_2$.
- Si $A \subseteq B$, área$(A) \leq$ área(B).
- Si $A \cap B = \emptyset$, área$(A \cup B) =$ área$(A) +$ área(B).

Consideremos una función $f : (a, b) \to \mathbb{R}$ en las mismas condiciones del apartado *i)* de la **Definición 14**. Admitiremos que la función $f(x)$ es positiva en todo el intervalo (a, b). Para cada punto $x \in (a, b)$, llamemos $A(x)$ al área encerrada por la función y las rectas verticales que pasan por el punto inicial a y por x.



Es evidente que para cada valor de $x \in [a, b)$, el área $A(x)$ será diferente, además, podemos aceptar que $A(a) = 0$ ya que cuando $x = a$ la anchura de la banda considerada será nula y que $A(b)$ es el área total sobre el tramo (a, b). En definitiva, A es una función que depende de x definida en el intervalo $[a, b]$,

$$\begin{aligned} A : [a, b] &\to \mathbb{R} \\ A(x) &= \text{área encerrada por } f \text{ en el tramo } [a, x]. \end{aligned}$$

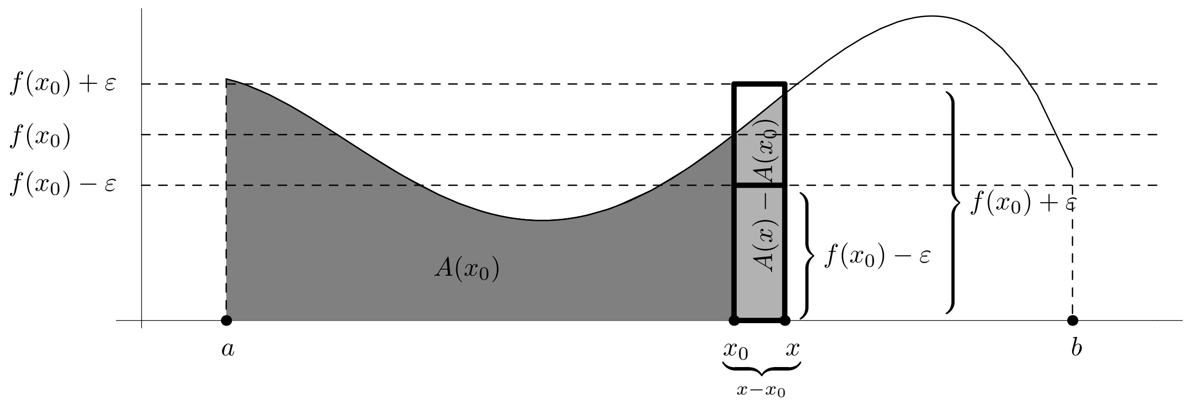
Tomemos un punto $x_0 \in [a, b]$ y comprobemos si la función $A(x)$ es derivable en ese punto. Para ello, debemos estudiar el límite

$$\lim_{x \to x_0} \frac{A(x) - A(x_0)}{x - x_0}.$$

Puesto que la función f es continua, tenemos que

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Entonces, elegido $\varepsilon \in \mathbb{R}^+$, es posible encontrar un tramo a izquierda y derecha de x_0, digamos $(x_0 - \delta, x_0 + \delta)$, en el que los valores de la función no se salen de la banda marcada por el intervalo $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Tomemos $x \in (x_0 - \delta, x_0 + \delta)$ dentro de ese tramo y observemos la gráfica correspondiente,



Es evidente que $A(x_0)$ es el área del tramo de la gráfica con sombreado más oscuro. Por su lado, $A(x)$ corresponde al área del tramo oscuro junto con el área del tramo sombreado en color más claro. La diferencia entre $A(x)$ y $A(x_0)$ es justamente ese tramo sombreado en color claro que por tanto tendrá área igual a $A(x)-A(x_0)$. Es inmediato que dicho tramo está contenido en el rectángulo de base el intervalo (x_0,x) y de altura $f(x_0)+\varepsilon$ y que al mismo tiempo contiene al rectángulo con base (x_0,x) y altura $f(x_0)-\varepsilon$. Aplicando las propiedades que antes hemos fijado para el área tendremos que

$$\text{área}\begin{pmatrix}\text{rectángulo de}\\ \text{base } (x_0,x) \text{ y}\\ \text{altura } f(x_0)-\varepsilon\end{pmatrix} \leq \text{área}\begin{pmatrix}\text{zona de}\\ \text{sombreado}\\ \text{claro}\end{pmatrix} \leq \text{área}\begin{pmatrix}\text{rectángulo de}\\ \text{base } (x_0,x) \text{ y}\\ \text{altura } f(x_0)+\varepsilon\end{pmatrix}$$

$$\Downarrow$$

$$(f(x_0)-\varepsilon)(x-x_0) \leq A(x)-A(x_0) \leq (f(x_0)+\varepsilon)(x-x_0)$$

$$\Downarrow$$

$$f(x_0)-\varepsilon \leq \frac{A(x)-A(x_0)}{x-x_0} \leq f(x_0)+\varepsilon.$$

Puesto que esta misma demostración es válida para ε tan pequeño como deseemos, no es difícil deducir de aquí que

$$\lim_{x\to x_0} \frac{A(x)-A(x_0)}{x-x_0} = f(x_0).$$

Puesto que x_0 es cualquier punto de $[a,b]$, obtenemos dos conclusiones:

- La función $A(x)$, que mide al área en el tramo desde el punto a hasta el punto x, es derivable en $[a,b]$.
- La derivada de la función $A(x)$ es

$$A'(x) = f(x),\ x \in [a,b].$$

Por tanto, $A(x)$ es una primitiva de $f(x)$. Si aplicamos la **Definición 14**, sabemos que

$$\int_a^x f(x)dx = A(x) - A(a).$$

Puesto que $A(a)=0$ finalmente deducimos que

$$A(x) = \int_a^x f(x)dx$$

y la integral definida entre dos puntos proporciona el área encerrada por la función sobre el intervalo determinado por ellos.

Con ello hemos demostrado la propiedad que expusimos en el apartado 3.3.1:

Propiedad 25. *Dada la función $f : (a, b) \to \mathbb{R}$ en las condiciones del apartado ii) de la* **Definición 14**, *el área comprendida entre el eje $y = 0$, las rectas verticales $x = a$ y $x = b$ y la gráfica de la función f se calcula mediante la integral definida*

$$\int_a^b f(x)dx$$

de modo que si f es positiva en (a, b) dicha integral proporcionará el área con signo positivo y si f es negativa lo hará con signo negativo.

3.4.5 El interés variable compuesto continuamente

Ampliación de las aplicaciones de la integral definida. Página 97

En el apartado 3.3.1 expusimos cuál era la fórmula para calcular la función $P(t)$ que arroja el capital acumulado hasta el año t, suponiendo que tenemos cierto capital inicial P_0 en una cuenta que abona un interés variable $I(t)$ (en tantos por uno) en cada año t. Pretendemos ahora demostrar dicha fórmula.

Para comenzar, estudiaremos la relación que existe entre la función de capital $P(t)$ y la función de interés $I(t)$. Para ello, preguntémonos primero que interpretación tendría $P'(t)$. Sabemos que $P'(t)$ será la velocidad de crecimiento en el siguiente sentido

$$\left.\begin{array}{l} P(t) = \text{capital acumulado en euros} \\ t = \text{año} \end{array}\right\} \Rightarrow P'(t) = \text{ velocidad de crecimiento en } \frac{\text{euros}}{\text{año}}.$$

Por tanto, $P'(t)$ indica cuántos euros crece el capital en cuenta por año cuando estamos en el año t. Podemos deducir entonces que justamente en el año t el incremento de capital que recibiremos en la cuenta debido a los intereses será justamente $P'(t)$. De este modo,

- Capital en la cuenta en el año t: $P(t)$.
- Cantidad ingresada por intereses en el año t: $P'(t)$.

Pero si conocemos el capital en cuenta y la cantidad ingresada por intereses, es fácil calcular el tipo de interés según la fórmula,

$$\text{Tipo de interes en tanto por uno } = \frac{\text{capital ingresado por intereses}}{\text{capital inicial}}$$

En consecuencia,

$$\text{Interés recibido en el año } t = \frac{\text{capital ingresado por intereses en el año } t}{\text{capital inicial en el año } t} = \frac{P'(t)}{P(t)}.$$

Ahora bien, el interés recibido en el año t es $I(t)$ así que finalmente tenemos

$$\boxed{I(t) = \frac{P'(t)}{P(t)}}. \tag{3.3}$$

Esta última fórmula nos permitiría calcular la función $I(t)$ fácilmente a partir de la función de capital $P(t)$. Sin embargo, si lo que conocemos es la función de interés $I(t)$, para calcular el capital $P(t)$ deberemos recurrir a la integral definida como veremos a continuación.

Es sencillo aplicar la regla de la cadena para comprobar que

$$(\log(P(t)))' = \frac{P'(t)}{P(t)}$$

con lo que, empleando (3.3),

$$(\log(P(t)))' = I(t).$$

Supongamos que sabemos que en un instante inicial t_0, el capital en cuenta es P_0 (es decir, $P(t_0) = P_0$). Si ahora calculamos la integral definida entre t_0 y t, en la expresión anterior tendremos

$$\int_{t_0}^{t} (\log(P(t)))' \, dt = \int_{t_0}^{t} I(t)dt$$

y empleando el apartado *ii)* de la **Propiedad 15** obtenemos

$$\log(P(t)) - \log(P(t_0)) = \int_{t_0}^{t} I(t)dt.$$

Pretendemos despejar $P(t)$ así que tomaremos el número e elevado a ambos miembros de la igualdad en la forma

$$\underbrace{e^{\log(P(t))-\log(P(t_0))}}_{= \frac{e^{\log(P(t))}}{e^{\log(P(t_0))}} = \frac{P(t)}{P(t_0)}} = e^{\int_{t_0}^{t} I(t)dt}$$

y puesto que $P(t_0) = P_0$ llegamos a que

$$\frac{P(t)}{P_0} = e^{\int_{t_0}^{t} I(t)dt},$$

de donde finalmente obtenemos

$$\boxed{P(t) = P_0 e^{\int_{t_0}^{t} I(t)dt}}\ ,$$

que es la fórmula para el capital sujeto a un interés compuesto continuamente, variable, $I(t)$, (en tanto por uno) cuando el capital en el año inicial t_0 es P_0.