Chapter 2

Derivative of functions

The derivative is the mathematical formalization of the concept of velocity. Since we use functions to represent phenomena that evolve with respect to time, the derivative will be fundamental for analyzing different aspects of these phenomena. In this topic, we present the fundamental aspects of differential calculus.

2.1 Concept of Derivative

The model that offers the clearest vision of the meaning of derivatives is probably the physical model of a body's velocity. Suppose we travel from city A to city B using a car. Let $e(t)$ be the number of kilometers we have traveled during the first t hours. If our journey lasts b hours, for each instant of time, t, between 0 and b, we will have a value for the function e. In other words, we have a function

$$\begin{array}{rcl} e : [0, b] & \to & \mathbb{R} \\ t & \mapsto & e(t) = \text{ Kilometers traveled up to hour } t \end{array}$$

Let's take an instant t_0 and suppose that from the information provided by the function e we aim to calculate the speed at which we were traveling exactly at that instant t_0. If we perform the calculation

$$\frac{e(b) - e(t_0)}{b - t_0}$$

we will obtain the average speed over the time interval $[t_0, b]$. But this does not provide us with the data we are looking for since the speed has not been constant during that period of time. We could take a smaller period as a reference, say from t_0 until a certain later moment, t. However, no matter how close we take t to t_0, the quotient

$$\frac{e(t) - e(t_0)}{t - t_0}$$

will only be the average of the speeds in the interval (t_0, t) which at best will serve to approximate the value of the exact data we are looking for. In reality, as t approaches t_0 we get answers closer and closer to the real one but never exact. The correct answer would be obtained if we could take t_0 equal to t so that the reference interval (t_0, t) would not introduce errors. It is evident that the latter is not possible but, instead, we can calculate

$$\lim_{t \to t_0} \frac{e(t) - e(t_0)}{t - t_0}.$$

This limit provides us with the instantaneous velocity exactly at the instant t_0 that we wanted to find. This limit is what we will call the derivative of the function e at the point t_0.

With this idea in mind, let's now see a formal definition of the derivative for a function at a point.

Definition 52. Given $f : D \to \mathbb{R}$:

i) We say that f is differentiable at $x_0 \in Ac(D)$ and that its derivative at that point is $L \in \mathbb{R}$ if the following limit exists:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L.$$

In that case, we will write

$$f'(x_0) = L \quad \text{or} \quad \frac{df}{dx}(x_0) = L.$$

If the aforementioned limit does not exist or is $\pm\infty$, we say that the function f is not differentiable at the point x_0.

ii) Given a subset $H \subseteq D$, we say that f is differentiable on H if it is differentiable at all points of H. If f is differentiable on D (on its entire domain) we say that f is a differentiable function.

iii) Let $f : D \to \mathbb{R}$ be a real function and let

$$D_1 = \{x \in D : f \text{ is differentiable at } x\}.$$

If $D_1 \neq \emptyset$, we call the derivative function of the function f the function

$$\begin{aligned} f' : D_1 &\to \mathbb{R} \\ x &\mapsto f'(x) \end{aligned}.$$

It could be that the limit of the derivative does not exist but that the one-sided limits do exist. Then we will speak of one-sided derivatives of the function.

Definition 53.

i) We say that the function f is differentiable from the left at x_0 and that the value of the left-hand derivative of f at x_0 is $L \in \mathbb{R}$, if the following limit makes sense and exists:

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = L.$$

ii) We say that the function f is differentiable from the right at x_0 and that the value of the right-hand derivative of f at x_0 is $L \in \mathbb{R}$, if the following limit makes sense and exists:

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = L.$$

As is the case for continuity, a function can only be differentiable at those points where it is defined.

Example 54. We know that, given the real numbers a and b, a function of the type

$$f(x) = ax + b$$

is a polynomial of degree 1 that is always represented as a straight line. Let's see how the calculation of the derivative for a straight line is performed without major difficulty by directly applying the definition.

Indeed, let's take any point $x_0 \in \mathbb{R}$ and try to calculate the derivative of the function f at x_0, $f'(x_0)$. According to the definition we have that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax + b - (ax_0 + b)}{x - x_0} = \lim_{x \to x_0} \frac{a(x - x_0)}{x - x_0} = \lim_{x \to x_0} a = a.$$

Therefore, a straight line is always a function differentiable at all points and the value of its derivative is always equal to the coefficient accompanying the variable, x, in the formula of the line. The derivative function will be

$$f'(x) = a.$$

For example, the derivative of the line $f(x) = 10x - 7$ is the constant function

$$f'(x) = 10.$$

Given a function $f : D \to \mathbb{R}$, if we assume it is differentiable at $x_0 \in D$ then we will have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \in \mathbb{R}.$$

But then,

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} (x - x_0) \frac{f(x) - f(x_0)}{x - x_0} = \underbrace{\lim_{x \to x_0} (x - x_0)}_{=0} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 0 \cdot L = 0$$

From which

$$\lim_{x \to x_0} f(x) = f(x_0).$$

In other words, we have the following:

Property 55. *Every function differentiable at a point is also continuous at that point.*

2.1.1 Interpretations of the Derivative

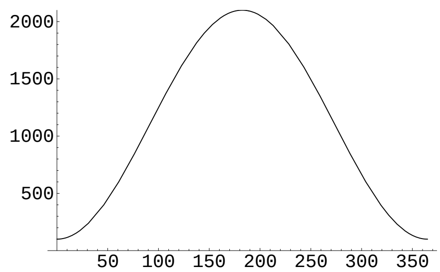
There are two fundamental interpretations of the concept of derivative. The first, as the rate of change of a certain quantity, has already been introduced in the introductory example. The second interpretation we will see is of a geometric nature.

The Derivative as a Rate of Change: The derivative of a function at a point is the rate of change or speed of the function at that point. We have already justified this interpretation in the initial example of the physical model. We will see the next two models in which the derivative is again interpreted as the rate of growth of a certain quantity represented by a function.

Examples 56.
1) The number of employees in a certain company is given, as a function of the number of days passed since the beginning of the year, by the function:

$$\begin{array}{c} f : [0, 365] \to \mathbb{R} \\ f(x) = 1100 - 1000 \cos\left(\frac{2\pi x}{365}\right) \end{array}.$$

The corresponding graph is,



We know that the derivative of the function f is the rate of change of $f(x)$ with respect to x. We have:

$$\left.\begin{array}{l} f(x) = \text{number of employees} \\ x = \text{days} \end{array}\right\} \Rightarrow f'(x) = \frac{\text{employees}}{\text{day}}.$$

Therefore, $f'(x)$ is the variation in employees (hirings or firings) per day on day x.

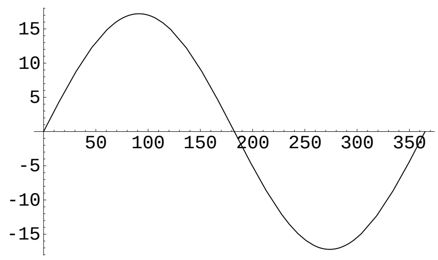
The derivative function of f is

$$f'(x) = \frac{400}{73}\pi \text{sen}\left(\frac{2\pi x}{365}\right).$$

Then, for example:

- $f'(50) = 13.0536 \Rightarrow$ On day $x = 50$, the hiring rate was 13.0536 employees/day. Therefore, on day 50, approximately 13.0536 employees were hired.
- $f'(182) = 0.148163 \Rightarrow$ On day $x = 182$, the hiring rate was 0.148163 employees/day. Therefore, on day 182, approximately 0.148163 employees were hired.
- $f'(260) = -16.7342 \Rightarrow$ On day $x = 260$, the hiring rate was -16.7342 employees/day. Therefore, on day 260, approximately 16.7342 firings occurred.

The graph of the derivative function allows us to get an idea of the evolution of the number of employees hired daily:



Note that during the first 182 days, the derivative function $f'(x)$, which measures daily hirings, is positive, which implies that each day new contracts are made and consequently we observe that the function $f(x)$ is increasing in that same period. At the same time, from day 182 onwards, the function of daily hirings given by the derivative $f'(x)$ is negative, which means that during that period firings occur and we can see that in that same stretch the function $f(x)$ is decreasing.

2) When analyzing the activity of a business, it is common to study what is called the cost function, which measures the cost necessary to produce a certain quantity of units.

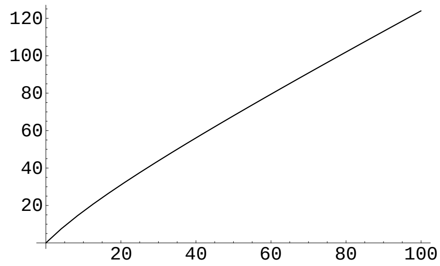
Suppose that in a company producing electronic components, the following cost function (measured in euros) is used:

$$C(x) = x + 10\log\left(\frac{x+10}{10}\right).$$

In that case:

- To produce $x = 10$ units the cost will be $C(10) = 16.9315$ euros.
- To produce $x = 20$ units the cost will be $C(20) = 30.9861$ euros.

The graph of the cost function is



If we calculate the derivative of the function $C(x)$ we will obtain the rate of change of the cost with respect to the quantity of units produced:

$$\left.\begin{array}{l} C(x) = \text{production cost in euros} \\ x = \text{units} \end{array}\right\} \Rightarrow C'(x) = \frac{\text{cost in euros}}{\text{unit}} = \frac{\text{euros}}{\text{unit}}.$$

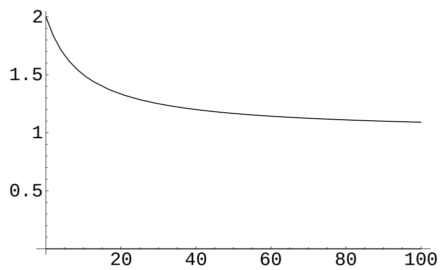
That is, $C'(x)$ is the cost per unit (what it costs to manufacture one unit) when we have produced x units. We have that

$$C'(x) = \frac{x+20}{x+10},$$

so:

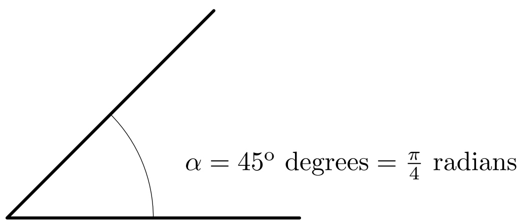
- The cost per unit when we have produced 10 units is $C'(10) = 1.5$ euros/unit.
- The cost per unit when we have produced 20 units is $C'(20) = 1.33$ euros/unit.

The derivative $C'(x)$ is called the marginal cost. In our case, the graph of marginal costs is:



It is observed that the price per unit when production is very low approaches 2 euros and as production increases it reduces to 1 euro.

Geometric Interpretation of the Derivative: Typically, an angle is measured in sexagesimal degrees or in radians. For example, in the following graph we observe an angle of 45^0 or, equivalently, $\frac{\pi}{4}$ radians:



However, on different occasions, the concept of slope is used to indicate or measure the value of an angle. Let's see its definition:

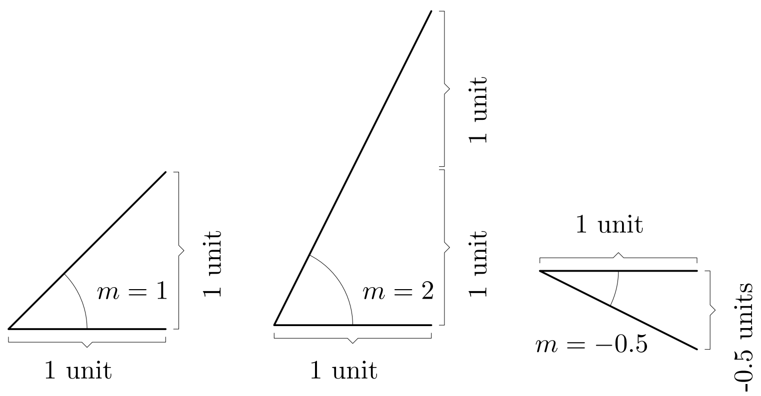
Definition 57. Given the angle α, we call the slope of α the number

$$m = \tan(\alpha).$$

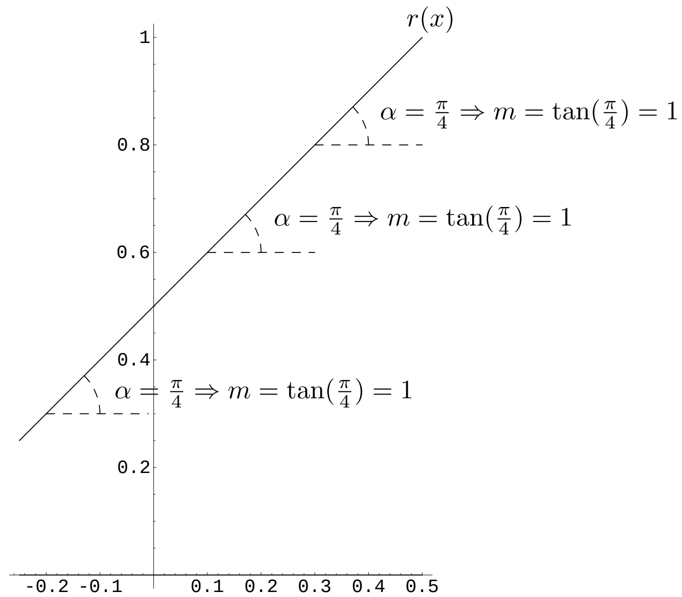
Thus, the slope of the angle in the graph will be

$$m = \tan(45^o) = \tan(\frac{\pi}{4}) = 1.$$

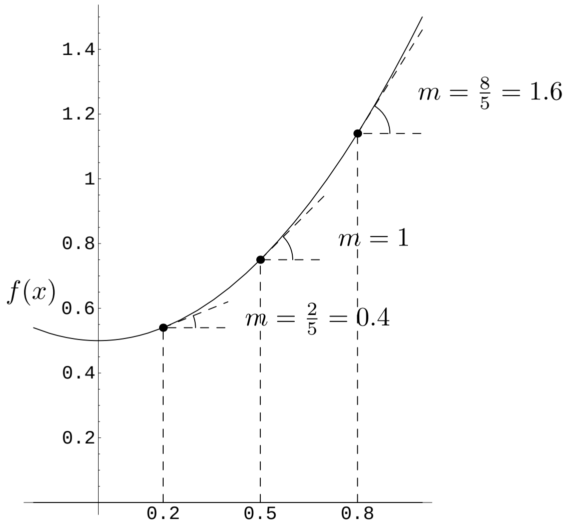
More intuitively, the slope of an angle tells us the units we ascend for each unit we advance if we follow the direction of that angle. For example, we represent below angles with slopes $m = 1$, $m = 2$ and $m = -0.5$.



If we consider a straight line, for example the function $r(x) = x + 1$, we can also calculate the angle that this line forms with the horizontal and its slope. Furthermore, it is evident that this angle will be the same at any point on the line:



On the other hand, if instead of a straight line we consider any function $f(x)$, we can verify that, at each point, the angle that the function forms with the horizontal is different and therefore we will also have a different slope at each point. In the following graph we observe how the function $f(x) = x^2 + \frac{1}{2}$ has different slopes at different points:



A reasoning similar to the one we performed at the beginning of the topic allows us to demonstrate that the slope of the angle that the function forms with the horizontal at each point is equal to the value of the derivative of the function at that point. That is, if f is differentiable at x_0 we will have,

$$\text{slope of } f \text{ at } x_0 = f'(x_0).$$

It is not difficult to verify that for the function in the previous example the derivative is

$$f'(x) = 2x.$$

In the graph we observe the slope of the function at the points $x_0 = 0.2$, $x_1 = 0.5$ and $x_3 = 0.8$. If we calculate the derivative at those points we will verify that, in each case, it coincides with the slope we see in the graph:

$$f'(0.2) = 0.4, \quad f'(0.5) = 1, \quad f'(0.8) = 1.6.$$

If we consider any straight line,

$$f(x) = ax + b,$$

we know that it has the same slope at all points and in fact, if we calculate its derivative, we observe that it always has the same value,

$$f'(x) = a$$

which is the slope of the line. In this way, we directly observe that the slope of any straight line is the coefficient, a, that accompanies the variable x in the formula of the line.

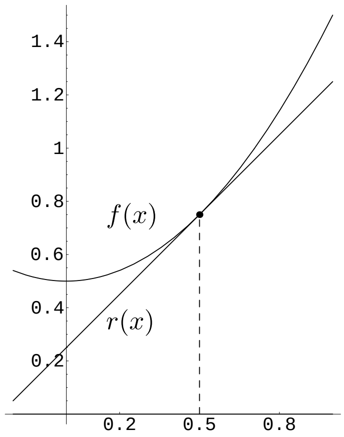
Example 58. The slope of the line $f(x) = 3x - 10$ is $m = 3$ since the coefficient accompanying the variable, x, is precisely 3. In another way we also have that $f'(x) = 3$.

The tangent line to a function f at the point x_0 is called the line that at the point x_0 takes the same value and has the same slope as f. In other words, the tangent line is the one that passes through the point $(x_0, f(x_0))$ in the same direction as f. Since we know that the slope of f at x_0 is $f'(x_0)$, it is easy to verify that the equation of the tangent line is

$$r(x) = f(x_0) + f'(x_0)(x - x_0).$$

For example, the tangent line to the function $f(x) = x^2 + \frac{1}{2}$ at the point $x_0 = 0.5$ will be

$$r(x) = f(0.5) + f'(0.5)(x - 0.5) \Rightarrow r(x) = 0.75 + 1 \cdot (x - 0.5) \Rightarrow r(x) = x + 0.25.$$



2.2 Calculation of Derivatives

Similarly to what we did for the limits, we will study the computation of the derivative in two stages for two groups of functions:

1. elementary functions,
2. functions obtained by composition or operation of elementary functions.

In the previous chapter, we saw that these two groups of functions, which always have a single formula, are always continuous. We now need two groups of properties that allow us to study differentiability and perform the calculation of the derivative, first, for the functions in inside item 1. and, then, for those included in section 2.

Property 59 (Derivative of Elementary Functions)**.**

1. *Given* $k \in \mathbb{R}$ *and* $f(x) = k$ *(constant function),* $f'(x) = 0, \ \forall x \in \mathbb{R}$.
2. *Given* $\alpha \in \mathbb{R}$, $f(x) = x^\alpha$ *is differentiable at all points where its derivative function,*
$$f'(x) = \alpha x^{\alpha - 1}.$$
is defined.
3. $(e^x)'(x) = e^x, \ \forall x \in \mathbb{R}$.
4. *For* $a > 0$, $(a^x)'(x) = \log(a) \cdot a^x, \ \forall x \in \mathbb{R}$.
5. $(\log)'(x) = \dfrac{1}{x}, \ \forall x \in \mathbb{R}^+$.
6. *For* $a \neq 1$, $(log_a)'(x) = \dfrac{1}{\log(a)} \cdot \dfrac{1}{x}, \ \forall x \in \mathbb{R}^+$.
7. $(\cos)'(x) = -sen(x), \ \forall x \in \mathbb{R}$.
8. $(sen)'(x) = \cos(x), \ \forall x \in \mathbb{R}$.
9. $(\tan)'(x) = \dfrac{1}{\cos^2(x)}, \ \forall x \in \mathbb{R} - \{\frac{\pi}{2} + k \cdot \pi / k \in \mathbb{Z}\}$.
10. $(arccos)'(x) = \dfrac{-1}{\sqrt{1 - x^2}}, \ \forall x \in (-1, 1)$.

11. $(arcsen)'(x) = \dfrac{1}{\sqrt{1-x^2}}$, $\forall x \in (-1, 1)$.

12. $(arctg)'(x) = \dfrac{1}{1+x^2}$, $\forall x \in \mathbb{R}$.

Let us now see another set of properties that allow us to study the second group, mentioned before, of functions obtained by operation or composition of elementary functions.

Properties 60. *Let f and g be real functions of a real variable. Then*

- *If f and g are differentiable at $x \in \mathbb{R}$ it holds that*
  1. *$f + g$ is differentiable at x and*
  $$(f+g)'(x) = f'(x) + g'(x).$$
  2. *$k \cdot f$ is differentiable at x for any constant $k \in \mathbb{R}$ and*
  $$(k \cdot f)'(x) = k \cdot f'(x).$$
  3. *$f \cdot g$ is differentiable at x and*
  $$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$
  4. *If $g(x) \neq 0$ then $\dfrac{f}{g}$ is differentiable at x and*
  $$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}.$$
- *(Chain rule) If f is differentiable at x and g is differentiable at $f(x)$ then $g \circ f$ is differentiable at x and it holds that:*
  $$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$
  The above formula is known as the chain rule.
- *(Inverse function theorem) If f is differentiable at x, bijective over its image and satisfies $f'(x) \neq 0$ then the inverse function of f, f^{-1}, is differentiable at $y = f(x)$ and it holds that*
  $$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}.$$

2.2.1 Differentiation of Piecewise Defined Functions

With the above, we can study the derivative of elementary functions or those obtained by operation or composition of them. The case of piecewise defined functions remains pending. As for the continuity, the study and calculation of the derivative of this type of functions is also done in two steps, first analyzing the interior of the intervals where each formula acts and then the points of definition change.

Given

$$f(x) = \begin{cases} f_1(x) & \text{if } a_1 \underset{\leq}{<} x \underset{\leq}{<} a_2, \\ f_2(x) & \text{if } a_2 \underset{\leq}{<} x \underset{\leq}{<} a_3, \\ \vdots & \qquad \vdots \\ f_{k-1}(x) & \text{if } a_{k-1} \underset{\leq}{<} x \underset{\leq}{<} a_k, \end{cases}$$

we will study its derivative in the following steps:

1) In the interior of the defining intervals: If the functions $f_1(x)$, $f_2(x), \dots, f_{k-1}(x)$ are respectively differentiable in the intervals (a_1, a_2), $(a_2, a_3), \dots, (a_{k-1}, a_k)$, then $f(x)$ will be differentiable in those intervals and its derivative will be

$$f'(x) = \begin{cases} f_1'(x) & \text{si} \quad a_1 < x < a_2, \\ f_2'(x) & \text{si} \quad a_2 < x < a_3, \\ \vdots & \qquad \vdots \\ f_{k-1}'(x) & \text{si} \quad a_{k-1} < x < a_k. \end{cases}$$

Note that in all cases we write "<" and never "≤" since in this first step we only analyze the interior of the defining intervals, leaving the points of change of definition for the next step.

2) At the points of change of definition: The points of definition change are $a_2, a_3, \dots, a_{k-1}$. At each of them, a_i, $i = 2, 3, \dots, k-1$, the function $f(x)$ will be differentiable if the following two conditions are met:

a) $f(x)$ is continuous at a_i,

b) the limit $\lim_{x \to a_i} f'(x)$ exists and is a real number (we use here $f'(x)$ as calculated previously in step **1)**).

If both conditions are met then, furthermore, we will have that

$$f'(a_i) = \lim_{x \to a_i} f'(x).$$

If such limit does not exist or takes the value $\pm\infty$, or if $f(x)$ is not continuous at a_i, then $f(x)$ is not differentiable at a_i.

Example 61. Study the differentiability of the function

$$f(x) = \begin{cases} e^x - x & \text{si} \quad x \le 0, \\ \cos(x) & \text{si} \quad 0 < x \le \frac{\pi}{2}, \\ x - \frac{\pi}{2} & \text{si} \quad \frac{\pi}{2} \le x. \end{cases}$$

and compute its derivative.

We calculate the derivative of this function following the two steps outlined above.

1) In the interior of the definition intervals $(-\infty, 0)$, $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \infty)$ the function $f(x)$ is respectively determined by the functions e^x, $\cos(x)$ and $x - \frac{\pi}{2}$, all of them elementary and differentiable functions. Therefore, in those intervals we calculate the derivative of $f(x)$ simply by differentiating the corresponding formula:

$$f'(x) = \begin{cases} e^x - 1 & \text{si} \quad x < 0, \\ -\text{sen}(x) & \text{si} \quad 0 < x \le \frac{\pi}{2}, \\ 1 & \text{si} \quad \frac{\pi}{2} < x. \end{cases}$$

2) We now study the definition points $x_0 = 0$ and $x_0 = \frac{\pi}{2}$.

$\boxed{x_0 = 0}$ Let's see the two required conditions:

1. Continuity of $f(x)$ at $x_0 = 0$: we have that

$$\left.\begin{array}{l} \bullet \ f(0) = e^0 - 0 = 1, \\ \bullet \ \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} e^x - x = e^0 - 0 = 1, \\ \bullet \ \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \cos(x) = \cos(0) = 1 \end{array}\right\} \Rightarrow f(x) \text{ is continuous at } x_0 = 0.$$

2. Limit of $f'(x)$ at $x_0 = 0$: using the formulas we obtained in step **1)** for $f'(x)$,

$$\left.\begin{array}{l} \bullet \ \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} e^x - 1 = e^0 - 1 = 0, \\ \bullet \ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -\text{sen}(x) = -\text{sen}(0) = 0 \end{array}\right\} \Rightarrow \lim_{x\to 0} f'(x) = 0.$$

Therefore $f(x)$ is differentiable at $x_0 = 0$ with

$$f'(0) = \lim_{x\to 0} f'(x) = 0.$$

$\boxed{x_0 = \frac{\pi}{2}}$ We proceed now as before.

1. Continuity of $f(x)$ at $x_0 = \frac{\pi}{2}$:

$$\left.\begin{array}{l} \bullet \ f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0, \\ \bullet \ \lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^-} \cos(x) = \cos(\frac{\pi}{2}) = 0, \\ \bullet \ \lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} x - \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi}{2} = 0 \end{array}\right\} \Rightarrow f(x) \text{ is continuous at } x_0 = \frac{\pi}{2}.$$

2. Limit of $f'(x)$ at $x_0 = \frac{\pi}{2}$:

$$\left.\begin{array}{l} \bullet \ \lim_{x\to \frac{\pi}{2}^-} f'(x) = \lim_{x\to \frac{\pi}{2}^-} -\text{sen}(x) = -\text{sen}(\frac{\pi}{2}) = -1, \\ \bullet \ \lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{x\to \frac{\pi}{2}^+} 1 = 1 \end{array}\right\} \Rightarrow \text{there is no } \lim_{x\to \frac{\pi}{2}} f'(x)$$

Therefore $f(x)$ is not differentiable at $x_0 = \frac{\pi}{2}$.

2.3 Differentiation and Shape Properties of a Function

In **Definition 9** of Chapter 1 we presented several properties that affect the appearance of the graphical representation of a function. They were what we called shape properties of the function. In this section we will add some more shape properties and study techniques that allow us to determine which of them are satisfied by a given function. Let us also remember that shape properties have a local character so it is necessary to determine in which intervals they are fulfilled.

Definition 62. Let $f : D \to \mathbb{R}$ be a real function of a real variable and let $H \subseteq D$. Then:

- We say that f is increasing (resp. strictly increasing, decreasing, strictly decreasing, constant) on H if $f|_H$ is increasing (resp. strictly increasing, decreasing, strictly decreasing, constant).
- We say that f has an absolute maximum at the point $x_0 \in D$ if
$$f(x_0) \geq f(x), \ \forall x \in D.$$
- We say that f has an absolute minimum at the point $x_0 \in D$ if
$$f(x_0) \leq f(x), \ \forall x \in D.$$
- We say that f has a local maximum at the point $x_0 \in D$ if $\exists\, a, b \in \mathbb{R}$, $a < x_0 < b$, such that
$$f(x_0) \geq f(x), \ \forall x \in D \cap (a, b).$$
If the inequality in the definition holds strictly except at the point x_0 (the inequality holds not only for $\geq$ but also for $>$), we say that f has a strict local maximum at x_0.

- We say that f has a local minimum at the point $x_0 \in D$ if $\exists\, a, b \in \mathbb{R}$, $a < x_0 < b$, such that

$$f(x_0) \le f(x),\ \forall x \in D \cap (a, b).$$

If the inequality in the definition holds strictly except at the point x_0 (the inequality holds not only for $\le$ but also for $<$), we say that f has a strict local minimum at x_0.

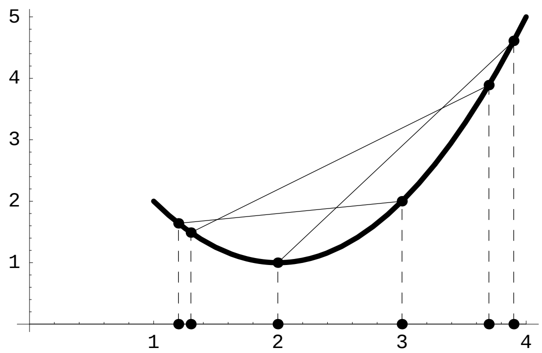
- We say that f is convex on H if $\forall a, b \in H$, such that $a < b$, it holds that

$$f(x) \le f(a) + \frac{x-a}{b-a}(f(b) - f(a)),\ \forall x \in H \cap (a, b),$$

that is, if within the set H, f is below the segment joining the points $(a, f(a))$ and $(b, f(b))$.

If the inequality in the definition holds strictly (the inequality holds not only with $\le$ but also with $<$) then we say that f is strictly convex on H.

Example 63. The equation $r(x) = f(a) + \frac{x-a}{b-a}(f(b) - f(a))$ is that of the line passing through the points $(a, f(a))$ and $(b, f(b))$. Therefore, a function is convex if the line joining two points of its graph is always above the function. In the following graph we represent a convex function. It can be observed how regardless of the choice we make for the points a and b, the line joining them is always above the function.



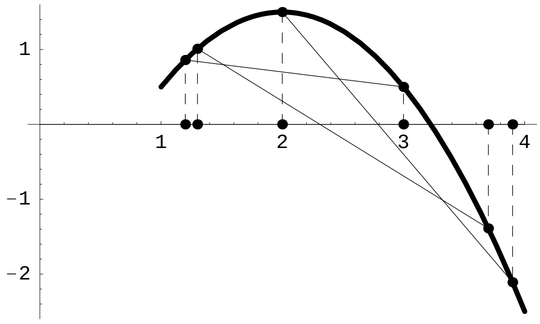
- We say that f is concave on H if $\forall a, b \in H$, such that $a < b$, it holds that

$$f(x) \ge f(a) + \frac{x-a}{b-a}(f(b) - f(a)),\ \forall x \in H \cap (a, b),$$

that is, if within the set H, f is above the segment joining the points $(a, f(a))$ and $(b, f(b))$.

If the inequality in the definition holds strictly (the inequality holds not only with $\ge$ but also with $>$) then we say that f is strictly concave on H.

Example 64. In the case of concavity, it is required that the line joining two points on the graph of the function is below the function ($f(x) \ge r(x) = f(a) + \frac{x-a}{b-a}(f(b) - f(a))$). The graphical situation is now

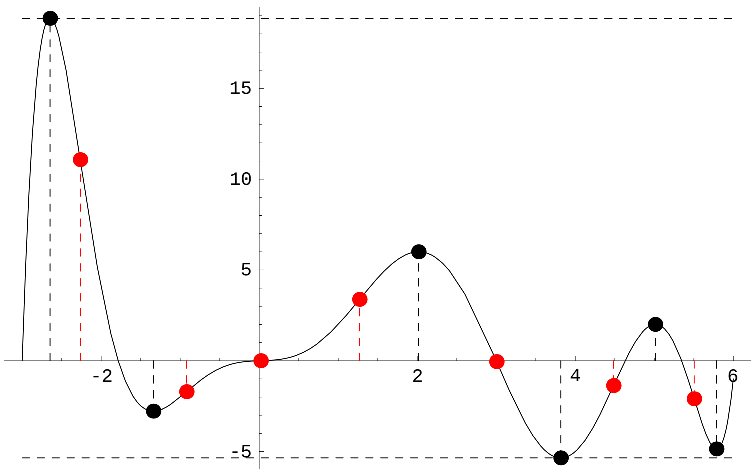


* We say that f has an inflection point at $x_0 \in D$ if $\exists\, a, b \in \mathbb{R}$, $a < x_0 < b$ such that one of the following conditions is met:

  a) f is strictly concave on $(a, x_0] \cap D$ and is strictly convex on $[x_0, b) \cap D$.
  b) f is strictly convex on $(a, x_0] \cap D$ and is strictly concave on $[x_0, b) \cap D$.

Example 65.

In the following image we represent the graph of a function $f : [-3, 6] \to \mathbb{R}$.



We have marked on the graph in black the points where relative maxima and minima are reached and in red the inflection points. It is evident that:

* Between two relative maxima/minima there is always an interval in which the function is monotonic (i.e., in which it is increasing or decreasing). These intervals are what are called intervals of increase and decrease of the function.
* Between each two inflection points, the function is either concave or convex. That is, the inflection points separate the different intervals of convexity and concavity.

The shape properties that a function satisfies in each interval will depend on the signs of its derivatives. The concept of successive derivative of a function comes into play here.

Definition 66.

Let $f : D \to \mathbb{R}$ be a real function. Then:

a) We define the first derivative function of f as the derivative function of f.

b) Given $n \in \mathbb{N}$, if the n-th derivative function of f is defined and is differentiable at some point, we define the $(n+1)$-th derivative function of f as the derivative function of the n-th derivative function of f.

If defined, we denote the n-th derivative function of f as

$$f^{n)} \qquad \text{or} \qquad \frac{d^n f}{dx^n}.$$

If, for $n \in \mathbb{N}$, the function $f^{n)}$ is defined at the point $x_0 \in D$, we say that the function f is n times differentiable at x_0 and we denote the value of the n-th derivative at that point as

$$f^{n)}(x_0), \qquad \frac{d^n f}{dx^n}(x_0) \quad \text{or} \quad \left.\frac{d^n f}{dx^n}\right|_{x=x_0}.$$

It is usually accepted that the 0-th derivative of a function is the function itself, that is,

$$f^{0)} = f.$$

The first, second and third derivative functions are usually denoted by f', f'' and f''', instead of $f^{1)}$, $f^{2)}$ and $f^{3)}$.

Definition 67. Given a set $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, we say that f is of class C^n on D if the following two conditions are met:

1. The function $f^{n)}$ is defined on all of D.
2. The function $f^{n)}$ is continuous on D.

The set of all functions of class C^n on D is denoted by $C^n(D)$.

Given $D \subseteq \mathbb{R}$ we denote by $C^0(D)$ or simply $C(D)$ the set of all continuous functions on D. Likewise, a function that is of class C^n on D for any $n \in \mathbb{N}$, is said to be a function of class C^∞ on D. The set of all functions of class C^∞ on D is denoted by $C^\infty(D)$.

Let us now see the criteria that allow us to discern which shape properties a function satisfies and where it satisfies them. As we indicated before, it will depend on the signs of the first three derivatives of the function.

Properties 68. *Let $f : D \to \mathbb{R}$ be a real function, let $I = (a, b) \subseteq D$ be an interval and $x_0 \in (a, b)$. It holds that:*

i) *If f is differentiable on I we have that:*

1. *If $f'(x) \geq 0,\ \forall x \in I$, then f is increasing on I.*
2. *If $f'(x) > 0,\ \forall x \in I$, then f is strictly increasing on I.*
3. *If $f'(x) \leq 0,\ \forall x \in I$, then f is decreasing on I.*
4. *If $f'(x) < 0,\ \forall x \in I$, then f is strictly decreasing on I.*

5. If $f'(x) = 0,\ \forall x \in I$, then f is a constant function on I.

ii) *If f is differentiable at x_0 and f has a local maximum or minimum at x_0 then $f'(x_0) = 0$.*

iii) *If f is of class C^2 on I and $f'(x_0) = 0$ then:*

1. If $f''(x_0) > 0$ then f has a strict local minimum at x_0.
2. If $f''(x_0) < 0$ then f has a strict local maximum at x_0.

iv) *If given $n \in \mathbb{N}$, f is of class C^n and it holds that*

$$f'(x_0) = f''(x_0) = \cdots = f^{n-2)}(x_0) = f^{n-1)}(x_0) = 0$$

and that

$$f^{n)}(x_0) \neq 0,$$

then:

1. If $f^{n)}(x_0) > 0$ and n is even then x_0 is a strict local minimum of f.
2. If $f^{n)}(x_0) < 0$ and n is even then x_0 is a strict local maximum of f.
3. If n is odd then f has neither a local maximum nor a local minimum at x_0.

v) *If f is of class C^2 on I then:*

1. If $f''(x) \geq 0,\ \forall x \in I$ then f is convex on I.
2. If $f''(x) > 0,\ \forall x \in I$ then f is strictly convex on I.
3. If $f''(x) \leq 0,\ \forall x \in I$ then f is concave on I.
4. If $f''(x) < 0,\ \forall x \in I$ then f is strictly concave on I.

vi) *If f is of class C^2 on I and x_0 is an inflection point of f then*

$$f''(x_0) = 0.$$

vii) *If f is of class C^3 on I and it holds that*

$$f''(x_0) = 0 \qquad \textit{and} \qquad f'''(x_0) \neq 0$$

then x_0 is an inflection point of f.

Example 69. Let us determine the shape properties of the function

$$f(x) = \frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{9}{2}x^2 + 18x + 1.$$

To do this, we begin by finding when the first derivative is zero:

$$f'(x) = x^3 - 8x^2 + 9x + 18 = 0 \Rightarrow \begin{cases} x = -1 \\ x = 3 \\ x = 6 \end{cases}.$$

The function $f'(x)$ is a polynomial and therefore is continuous in $\mathbb{R}$. If we check the sign of the value of $f'(x)$ at arbitrary points in the intervals $(-\infty, -1)$, $(-1, 3)$, $(3, 6)$ and $(6, \infty)$, a direct application of Bolzano's Theorem leads us to the conclusion that these signs will determine the sign at the rest of the points in each interval. Ultimately, we arrive at

$$\boxed{\begin{array}{l} f'(x) < 0,\ \forall x \in (-\infty, -1). \\ f'(x) > 0,\ \forall x \in (-1, 3). \\ f'(x) < 0,\ \forall x \in (3, 6). \\ f'(x) > 0,\ \forall x \in (6, +\infty). \end{array}}$$

To determine the intervals of convexity and concavity we calculate the second derivative,

$$f''(x) = 3x^2 - 16x + 9.$$

In this case,

$$f''(x) = 3x^2 - 16x + 9 = 0 \Rightarrow \begin{cases} x = \frac{1}{3}\left(8 - \sqrt{37}\right) \approx 0.639. \\ x = \frac{1}{3}\left(8 + \sqrt{37}\right) \approx 4.694. \end{cases}$$

and reasoning as before,

$$\boxed{\begin{array}{l} f''(x) > 0, \ \forall x \in \left(-\infty, \frac{1}{3}\left(8 - \sqrt{37}\right)\right). \\ f''(x) < 0, \ \forall x \in \left(\frac{1}{3}\left(8 - \sqrt{37}\right), \frac{1}{3}\left(8 + \sqrt{37}\right)\right). \\ f'(x) > 0, \ \forall x \in (\frac{1}{3}\left(8 + \sqrt{37}\right), \infty). \\ \text{In particular, } f(-1) > 0, \ f(3) < 0 \text{ and } f(6) > 0. \end{array}}$$

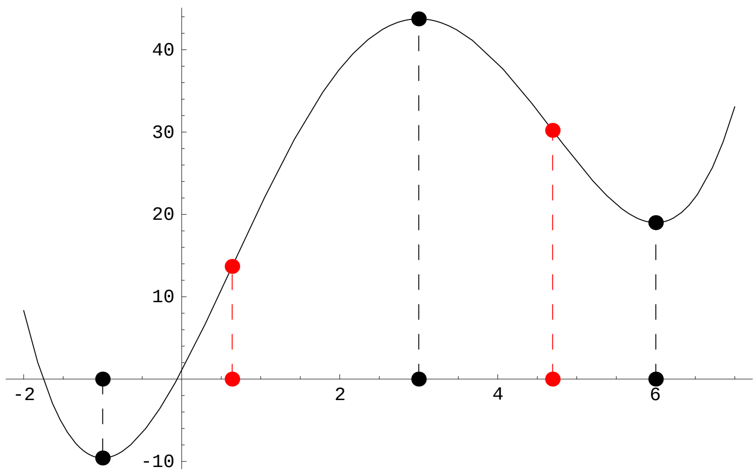
Finally, we have that $f'''(x) = 6x - 16$ so that

$$\boxed{f'''\left(\frac{1}{3}\left(8 - \sqrt{37}\right)\right) \neq 0 \text{ and } f'''\left(\frac{1}{3}\left(8 + \sqrt{37}\right)\right) \neq 0.}$$

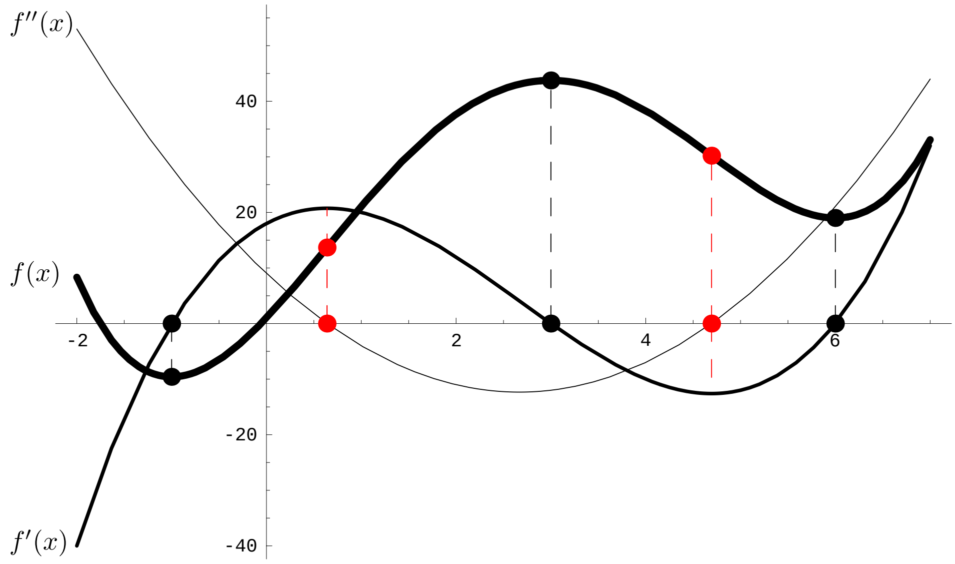
Taking into account the information we have boxed and **Properties 68** we have that:

- The function f is decreasing on the intervals $(-\infty, -1)$ and $(3, 6)$.
- The function f is increasing on the intervals $(-1, 3)$ and $(6, \infty)$.
- The function f has relative minima at the points $x = -1$ and $x = 6$.
- The function f has a relative maximum at the point $x = 3$.
- The function f is convex on the intervals $\left(-\infty, \frac{1}{3}\left(8 - \sqrt{37}\right)\right)$ and $\left(\frac{1}{3}\left(8 + \sqrt{37}\right), \infty\right)$.
- The function f is concave on the interval $\left(\frac{1}{3}\left(8 - \sqrt{37}\right), \frac{1}{3}\left(8 + \sqrt{37}\right)\right)$.
- The function f has inflection points at $\frac{1}{3}\left(8 - \sqrt{37}\right)$ and $\frac{1}{3}\left(8 + \sqrt{37}\right)$.

If we calculate the value of the function at the relative maximum and minimum points and at the inflection points, all the above information leads us to the following graph:



Representing together the function $f(x)$ and the derivatives $f'(x)$ and $f''(x)$ we graphically observe how the intervals of increase, decrease, concavity and convexity correspond to the signs of f' and f''.



2.4 L'Hôpital's Rule

In Chapter 1 we studied the case of different limits that could not be calculated by directly applying the algebraic properties of the limit since they lead to indeterminate forms. Given two functions, $f, g : D \to \mathbb{R}$, if $f(x_0) = g(x_0) = 0$ or $f(x_0) = g(x_0) = \infty$, the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

leads to an indetermination of the type $\frac{0}{0}$ or of the type $\frac{\infty}{\infty}$ that we only know how to solve in a couple of very specific cases. If $f(x_0) = g(x_0) = 0$, we could modify the way we have written the previous limit and propose the following chain of equalities:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\dfrac{f(x) - f(x_0)}{x - x_0}}{\dfrac{g(x) - g(x_0)}{x - x_0}} \boxed{=} \frac{\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}}{\lim\limits_{x \to x_0} \dfrac{g(x) - g(x_0)}{x - x_0}} = \frac{f'(x_0)}{g'(x_0)}.$$

We have boxed the second symbol $\boxed{=}$ to indicate that this step must be justified more carefully. In any case, this is the idea behind l'Hôpital's results for the calculation of limits of quotients, which we formulate precisely in the following property.

Theorem 70 (L'Hôpital's Rules). *Let f and g be two real functions, let $x_0 \in \mathbb{R}$ and an interval $I = (a, b) \subseteq \mathbb{R}$ such that $x_0 \in I$ so that the following conditions are verified*

1. *f and g are differentiable on $I - \{x_0\}$.*
2. *$g'(x) \neq 0,\ \forall x \in I - \{x_0\}$.*
3. *One of the following two conditions holds:*
   a) $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$.
   b) $\lim\limits_{x \to x_0} |g(x)| = +\infty$.

Then it holds that:

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = L,$$

where L can be a real number, $+\infty$ or $-\infty$. The property is also correct for the left-hand or right-hand limit.

Examples 71.
1) If we calculate the limit $\lim_{x\to 0} \frac{\text{sen}(x)}{x}$ directly,

$$\lim_{x\to 0} \frac{\text{sen}(x)}{x} = \frac{0}{0},$$

we obtain an indetermination form. The functions in the numerator and denominator are under the conditions of L'Hôpital's Rule so we can avoid the indetermination by differentiating both functions in the numerator and denominator,

$$\lim_{x\to 0} \frac{\text{sen}(x)}{x} = \lim_{x\to 0} \frac{\text{sen}'(x)}{(x)'} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$$

2) Let us calculate the limit $\lim\limits_{x\to 0} x^x$. Initially we have that

$$\lim_{x\to 0} x^x = 0^0.$$

But 0^0 is an indetermination and we cannot decide the value the limit will reach in this way. Using the properties of the logarithm we know that

$$x^x = e^{x\log(x)},$$

in which case, using the properties of the limit with respect to exponentiation, we can write,

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\log(x)} = e^{\lim_{x\to 0} x\log(x)}.$$

Now, instead of the initial limit, we must calculate $\lim_{x\to 0} x\log(x)$. But writing this product as a quotient and applying l'Hôpital's rule,

$$\lim_{x\to 0} x\log(x) = \lim_{x\to 0} \frac{\log(x)}{x^{-1}} = \lim_{x\to 0} \frac{\log'(x)}{(x^{-1})'} = \lim_{x\to 0} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0} -x = 0.$$

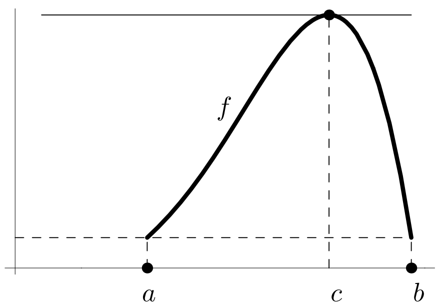
Finally,

$$\lim_{x\to 0} x^x = e^{\lim_{x\to 0} x\log(x)} = e^0 = 1.$$

2.5 Additional Material

2.5.1 Classical Theorems of Differentiation

Given a function $f : [a, b] \to \mathbb{R}$, if $f(a) = f(b)$, its graph will be something like



Intuition tells us that, necessarily, at some intermediate point between a and b the tangent to the function must be horizontal. Rolle's Theorem affirms that this is indeed the case.

Theorem 72 (Rolle's Theorem)**.** *Let f be a function continuous on an interval $[a, b]$ and differentiable on (a, b) then if $f(a) = f(b)$ it holds that there exists $c \in (a, b)$ such that*

$$f'(c) = 0.$$

When $f(a) \neq f(b)$ the previous reasoning is not valid but it is easy to formulate a version of Rolle's Theorem for this situation. If $f(a) = f(b)$, the line joining $(a, f(a))$ with $(b, f(b))$ is horizontal and the line whose existence is postulated by Rolle's Theorem must also be horizontal. What we have is that the tangent at some point is parallel to the line joining the initial points of the graph of the function. The Mean Value Theorem affirms that the latter is true even when $f(a) \neq f(b)$.

Theorem 73 (Mean Value Theorem)**.** *Let f be a function continuous on an interval $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

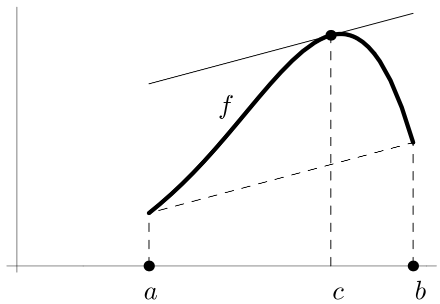
We know that the slope, m, of the line joining $(a, f(a))$ and $(b, f(b))$ is the tangent of the angle it forms with the horizontal and therefore

$$m = \frac{f(b) - f(a)}{b - a}.$$

On the other hand, at the point $c \in (a, b)$, the slope, m_1, of the tangent line will be $f'(c)$ and if we solve for it in the equality of the Mean Value Theorem we have that

$$m_1 = f'(c) = \frac{f(b) - f(a)}{b - a} = m.$$

In that case, the slope, m, of the line joining the initial and final points of the graph of f and the slope of the tangent line at point c coincide and both lines are parallel. As we have seen before, this constitutes a generalization of Rolle's Theorem to the case where $f(a) \neq f(b)$.



Rolle's Theorem is often used to prove that an equation has a unique solution in a certain interval. Suppose we are solving the equation

$$f(x) = 0$$

and we have found two solutions a and b for this equation. In that case we will have

$$f(a) = f(b) = 0$$

and if the function satisfies the hypotheses of Rolle's Theorem we will have that there exists an intermediate point between a and b such that

$$f'(c) = 0.$$

Now, if previously, by some means, we have verified that the function f' never vanishes, the last identity cannot be true, in which case the initial conjecture that we have two solutions of the equation $f(x) = 0$ cannot be correct, so there must be a unique solution.

Example 74. Let us show that the equation $e^x + x = 2$ has a unique solution. To do this, let us take the function $f(x) = e^x + x - 2$ and prove equivalently that $f(x) = 0$ has a unique solution.

Existence of solution (Bolzano's Theorem): The function $f(x)$ is continuous. If we find two points a and b where the function takes values with different signs, the Bolzano Theorem will guarantee the existence of a solution. Now, it is easy to check that

$$f(0) = e^0 + 0 - 2 = 1 - 2 < 0, \quad \text{and} \quad f(2) = e^2 + 2 - 2 = e^2 > 0.$$

Therefore, there must exist a solution, c, of $f(x) = 0$ which, moreover, will be in the interval $(0, 2)$ (we could approximate it by the bisection method).

Uniqueness of solution (Rolle's Theorem): We already know that $f(x) = 0$ has at least one solution which we have called c. Suppose we have another solution $c_1 \neq c$. Reasoning as we indicated before, since $f(c) = 0 = f(c_1)$, we can apply Rolle's Theorem and affirm that there exists ξ between c and c_1 such that

$$f'(\xi) = 0.$$

However, $f'(x) = e^x + 1$ and it is evident that

$$f'(x) > 0, \ \forall x \in \mathbb{R}.$$

Consequently, that second solution c_1 cannot exist. The only solution is the one we located before, c.

2.5.2 Taylor and McLaurin Series

Suppose we want to find a function such that at a certain point x_0 its derivatives successively take the values $f_0, f_1, \ldots, f_k$, that is,

$$f(x_0) = f_0, \ f'(x_0) = f_1, \ f''(x_0) = f_2, \ldots, f^{n)}(x_0) = f_n.$$

There are many ways to solve this problem but the simplest way is to consider the following polynomial:

$$p_n(x) = f_0 + \frac{f_1}{1!}(x - x_0) + \frac{f_2}{2!}(x - x_0)^2 + \cdots + \frac{f_n}{n!}(x - x_0)^n \tag{2.1}$$

is a solution to this problem, where $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ is what is called the factorial number. It is easy to check that

$$p(x_0) = f_0, \ p'(x_0) = f_1, \ p''(x_0) = f_2, \ldots, p^{n)}(x_0) = f_n.$$

Let us now take a function $f : D \to \mathbb{R}$, n times differentiable at a certain point x_0. Suppose we only have information about the function at the point x_0 we know the value of the function, $f(x_0)$, and that of its first n derivatives, $f'(x_0), f''(x_0), \ldots, f^{n)}(x_0)$? Will it be possible to reconstruct the function f from this information? It is evident that this is impossible but at least we can try to find a function as similar as possible to f. Since we only know the first derivatives of f, the only thing we can do is to obtain a function that coincides with f in those first derivatives and for that we can use the polynomial $p_n(x)$ from (2.1) for $f_0 = f(x_0), f_1 = f'(x_0), \ldots, f_n = f^{n)}(x_0)$. What we will obtain is, in a certain sense, the best approximation of f that we can calculate knowing only its first derivatives at the point x_0. That polynomial is what is called the Taylor Polynomial of the function f at x_0.

Definition 75. Let $f : D \to \mathbb{R}$ be a function of class C^n on D and let $x_0 \in D$. We call the Taylor polynomial of degree n of f at x_0 the polynomial

$$p_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{n)}(x_0)}{n!}(x - x_0)^n.$$

The Taylor polynomial of degree n at $x_0 = 0$ is also called the McLaurin polynomial of degree n.

The Taylor polynomial of a function constitutes an approximation of said function that has the advantage of being easier to handle. The original function, $f(x)$, could have a complicated expression but $p_n(x)$ is always a polynomial on which most calculations can be performed easily. Sometimes it will be possible to replace a function by its Taylor polynomial. However, when taking the Taylor polynomial instead of the function, we commit an error since the polynomial is not exactly equal to it. The following property gives us an expression for the error committed when making that substitution.

Property 76. *Let $I = (a, b)$ be an interval, let $f : I \to \mathbb{R}$ be a function of class C^{n+1} on I and let $x_0 \in I$. Then for any $x \in I$ there exists a real point ξ located between x_0 and x such that*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

The above formula is known as the Taylor formula of degree n of the function f at the point x_0.

The Taylor formula gives us the error committed when taking the Taylor polynomial instead of the function. Note that if we call $p_n(x)$ the Taylor polynomial of degree n of f at x_0, using the Taylor formula, we can write

$$f(x) - p_n(x) = \frac{f^{n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

and therefore the error committed will be

$$E_n(x) = \left| \frac{f^{n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \right|.$$