Chapter 1

Functions

At the base of many mathematical models lies the concept of a function. The description of a phenomenon that evolves with respect to time is generally done through a function $f(t)$ that at each instant t provides the number of individuals in a population, the size of a certain growing object, the income received in an account, etc.

As an example, suppose we are studying the population of a certain city where initially 1000 people live. If we denote by P the number of inhabitants in the city we could then write

$$P = 1000.$$

However, it is evident that the population of any city varies over time and that, due to births and deaths, the number of inhabitants will change from one year to the next. In such a case, the previous equality will not be correct and the population P will not be a fixed number but an expression $P(t)$ that will be different for each year t. In other words, we have a magnitude, the population P, that varies with respect to another, time t. The dependence between P and t is usually expressed by a mathematical formula. Thus, for example, after performing the corresponding analysis, suppose that, in the specific case of the city we are studying, we have

$$P(t) = 1000e^{0.1t}.$$

Then, using this formula, we can easily calculate the number of inhabitants for any year by simply substituting t with the appropriate value. For example:

- in year $t = 1$ the population is $P(1) = 1000e^{0.1\cdot 1} = 1105.17$,
- in year $t = 4$ the population is $P(4) = 1000e^{0.1\cdot 4} = 1491.82$.

On the other hand, in general, studies are conducted for a certain period or time interval so that the previous formula will be effective only for a specific range of years. That is, the value of t will be within certain limits. If, for example, the study was conducted for the first ten years, the value of t will be between 0 and 10, within what we will later call the interval $[0, 10]$ and we will indicate this by completing the information we gave earlier as follows,

$$\begin{array}{c} P : [0, 10] \to \mathbb{R} \\ P(t) = 1000e^{0.1t} \end{array}.$$

This is what we call a mathematical function and the elements that appear in this expression provide all the information we need about it:

- $[0, 10]$ is the interval within which the variable moves. Therefore, in this case, the formula will be valid from year $t = 0$ to year $t = 10$.
- P is the name of the function and t is its variable. P, is the population that depends on time, t, which we indicate by writing $P(t)$ (P is a function of t).

- $P(t) = 1000e^{0.1t}$ is the formula that determines how P depends on t. Using this formula, once we know t we can calculate P.

In what follows, we will see each of these elements in more detail. We will begin by studying what an interval is and then we will see the precise definition of a function to continue studying the mathematical formulas that have application in our discipline.

1.1 The Real Line. Intervals

All measurements of magnitudes and phenomena we encounter in the real world are usually represented by numbers. Numbers are the basis for indicating how long a phenomenon lasts, the length of an object, and in general for measuring and describing the physical properties of anything or event. In mathematics, many different types of numbers are used: positive, negative, fractional or integer, with or without decimals. But, in any case, the numbers we use to describe the magnitudes of the real world admit a decimal expression of the form

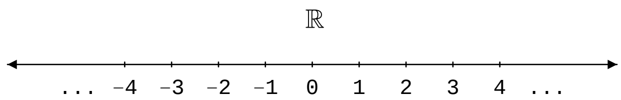
$$\underbrace{\pm}_{\text{sign}}\underbrace{eee\ldots e}_{\substack{\text{integer}\\ \text{part}}}.\underbrace{ddd\ldots}_{\substack{\text{fractional}\\ \text{part}}},$$

where $eee\ldots e$ and $ddd\ldots$ are lists of digits of variable length (the part $ddd\ldots$ could have infinitely many digits) that represent, respectively, the integer and fractional parts of the number. In the usual notation we can omit some elements and thus, for positive numbers, the + sign is not necessary to include or, when all the decimal digits are zero, neither they nor the decimal point are written. For example -321.1234322, 7789.45433, 34543 are examples of numbers that in more detail we could write as

$$\underbrace{-}_{\text{sign}}\underbrace{321}_{\substack{\text{integer}\\ \text{part}}}.\underbrace{1234322}_{\substack{\text{fractional}\\ \text{part}}}, \quad \underbrace{7789}_{\substack{\text{integer}\\ \text{part}}}.\underbrace{45433}_{\substack{\text{fractional}\\ \text{part}}}, \quad \underbrace{34543}_{\substack{\text{integer}\\ \text{part}}}$$

These numbers with a sign, integer part, and fractional part are the ones we use to measure the magnitudes we observe in the real world and, therefore, are called real numbers. There are infinitely many real numbers and all of them grouped together form the set of real numbers which is denoted as $\mathbb{R}$.

The set $\mathbb{R}$ can be represented by a line in the form



In this representation, numbers with decimals occupy intermediate positions between the integers so that the final graph of all the numbers of $\mathbb{R}$ completes the entire line, which is why the set $\mathbb{R}$ is also called the real line.

In most examples we will not work with the entire line $\mathbb{R}$ since in many cases we will be interested in focusing only on a fragment or interval of it. Thus, for example, in the previous example, we studied the population only from year 0 to 10, so we did not consider all the numbers of $\mathbb{R}$ but only those between these two values. The way we have to indicate which segment of $\mathbb{R}$ we are working with is by using intervals.

Definition 1. Let $a, b \in \mathbb{R}$. Then:

- We call open intervals the subsets of $\mathbb{R}$,

$$(a,b) = \{x \in \mathbb{R}/a < x < b\}, \quad (a,+\infty) = \{x \in \mathbb{R}/a < x\} \quad \text{and} \quad (-\infty,b) = \{x \in \mathbb{R}/x < b\}.$$

  We will say that the numbers a and/or b are the endpoints of such intervals.

- We call closed intervals the subsets of $\mathbb{R}$,

$$[a,b]=\{x\in\mathbb{R}/a\le x\le b\},\quad [a,+\infty)=\{x\in\mathbb{R}/a\le x\}\quad\text{and}\quad(-\infty,b]=\{x\in\mathbb{R}/x\le b\}.$$

  We will say that the numbers a and/or b are the endpoints of such intervals.

- We call half-open or half-closed intervals the subsets of $\mathbb{R}$,

$$[a,b)=\{x\in\mathbb{R}/a\le x<b\}\quad\text{and}\quad(a,b]=\{x\in\mathbb{R}/a<x\le b\}.$$

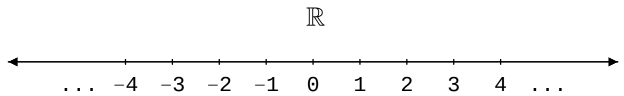
  We will say that the numbers a and b are the endpoints of such intervals.

Remark. We will use the following notations:

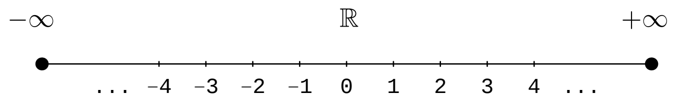
$$\begin{aligned}\mathbb{R}^+&=(0,+\infty)=\text{set of all positive real numbers.}\\ \mathbb{R}^+_0&=[0,+\infty)=\mathbb{R}^+\cup\{0\}.\\ \mathbb{R}^-&=(-\infty,0)=\text{set of all negative real numbers.}\\ \mathbb{R}^-_0&=(-\infty,0]=\mathbb{R}^-\cup\{0\}.\end{aligned}$$

Sometimes the notation $(-\infty,+\infty)$ is also used to refer to the entire set $\mathbb{R}$. Also, given $a\in\mathbb{R}\cup\{-\infty,+\infty\}$ we will accept that $(a,a)=\emptyset$.

In the previous notation we use the symbols $+\infty$ and $-\infty$ to refer to 'plus infinity' and 'minus infinity'. Neither $-\infty$ nor $+\infty$ are numbers (they are not elements of $\mathbb{R}$) and we will always use them as part of the mathematical notation. However, it is possible to conceive an intuitive image of the meaning of $-\infty$ or $+\infty$. To do this, it is enough to remember that the graphical representation of the set $\mathbb{R}$ of real numbers is a line that extends unlimitedly in both directions



Although we have just said that the real line is unlimited, it is possible to imagine that if we managed to traverse it to the end, both in one direction and the other, we would reach its extremes so that $-\infty$ would be the point located at the left extreme and $+\infty$ the one at the right



This image of $-\infty$ and $+\infty$ as the extremes of the real line is useful for understanding several concepts throughout this topic and the following ones.

Definition 2. Given $x\in\mathbb{R}$ we call the absolute value of the number x, and denote it $|x|$, the number defined by

$$|x|=\begin{cases}-x, & \text{if } x<0,\\ x, & \text{if } 0\le x.\end{cases}$$

Properties 3. $\forall a,b\in\mathbb{R}$

1. $0\le|a|$.
2. $|a+b|\le|a|+|b|$.
3. $\big||a|-|b|\big|\le|a-b|$.
4. *Given* $r\in\mathbb{R}^+_0$ *the following equivalence holds*

$$|a|\le r\quad\Leftrightarrow\quad -r\le a\le r.$$

Remark. Given any interval with endpoints a and b the length of the interval will be precisely the distance between point a and b which is precisely the absolute value of the difference between b and a, that is $|b-a|$. For example,

$$\begin{aligned}
\text{length of } [3,9) &= \mathbf{d}(3,9) = |9-3| = 6,\\
\text{length of } [-7,-2] &= \mathbf{d}(-7,-2) = |-2-(-7)| = 5,\\
\text{length of } [12,\infty) &= \mathbf{d}(12,\infty) = |\infty - 12| = \infty.
\end{aligned}$$

1.2 Real Functions.

A function is a particular case of a mapping in which the domain and codomain sets are subsets of $\mathbb{R}$.

Definition 4. A real function of a real variable is a mapping

$$f : D \to \mathbb{R},$$

where $D \subseteq \mathbb{R}$ is a subset consisting of real numbers that we call the domain of the function.

Generally, real functions will be defined by a formula or set of formulas and can be expressed in one of the following two ways:

1. Using the usual notation for mappings between sets: For example, the function

$$\begin{aligned} f : (-1,1) &\longrightarrow \mathbb{R} \\ f(x) &= \frac{1}{1-x^2} \end{aligned}$$

is expressed using the notation for mappings between sets. By observing its definition we can obtain the following information about f:

- ⋆ <u>the defining formula</u>: $\dfrac{1}{1-x^2}$.
- ⋆ <u>the domain of the function</u>: the interval $(-1,1)$.

2. Using only the defining formula: In this case, only the formula that defines the function will be indicated. Thus, for example, the function g given by

$$g(x) = \frac{1}{1-x^2},$$

is a real function that has been defined by indicating only its formula. Therefore, we do not know the domain of g as it is not explicitly given and we have to compute it following the following rule:

> In this case, when we only have the defining formula, the domain of the function will be the set of real numbers for which the formula makes sense (for which it is possible to calculate the value of the function at those points).

For the function g that we have taken as an example, from the definition we have given for it we obtain the following information:

- ⋆ <u>the defining formula</u>: $\dfrac{1}{1-x^2}$.
- ⋆ <u>the domain of the function</u>: It will be the set of real numbers for which the defining formula makes sense, that is, for which the formula can be applied obtaining a real value. Note that for this function we have that:

- for $x = 1$, $g(1) = \dfrac{1}{1-1^2} = \dfrac{1}{0}$ an expression that makes no sense and therefore $g(1)$ cannot be calculated.
- for $x = -1$, $g(-1) = \dfrac{1}{1-(-1)^2} = \dfrac{1}{0}$ so that $g(-1)$ cannot be calculated.
- for $x \in \mathbb{R} - \{1, -1\}$ it will always be possible to calculate the value of $g(x)$.

Therefore the domain of g will be the set

$$D = \mathbb{R} - \{-1, 1\}.$$

Note that the functions

$$\begin{array}{c} f : (-1,1) \longrightarrow \mathbb{R} \\ f(x) = \dfrac{1}{1-x^2} \end{array} \quad \text{and} \quad g(x) = \frac{1}{1-x^2}$$

have the same formula (namely, $\frac{1}{1-x^2}$) but are not the same function. For f the domain is the interval $(-1, 1)$ while for g it is $\mathbb{R} - \{-1, 1\}$. Since the domain of the function f is the set $(-1, 1)$ we will only be able to calculate the values of f for points in that set, that is, for points between -1 and 1; in this way, if we are asked to calculate $f(4)$ we must respond that the function f is not defined at the point 4 since $4 \notin (-1, 1)$ and is not in the domain. In contrast, note that it is indeed possible to calculate $g(4)$.

We observe in this way that for two functions to be equal it is not sufficient that they have the same defining formula. They must also have the same domain.

1.2.1 Graphical Representation

It is essential to understand the methods for representing functions. On the graph of a function we can immediately appreciate different properties that we would not notice if we only have the formula.

Mathematical functions are represented on a plane with two perpendicular axes. The horizontal axis will correspond to the variable and the vertical one to the values taken by the function. In this way, the graphic representation of is obtained by representing the points of the set called graph of the function.

Definition 5. Given a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$,

$$\text{graph}(f) = \{(x, f(x)) : x \in D\}.$$

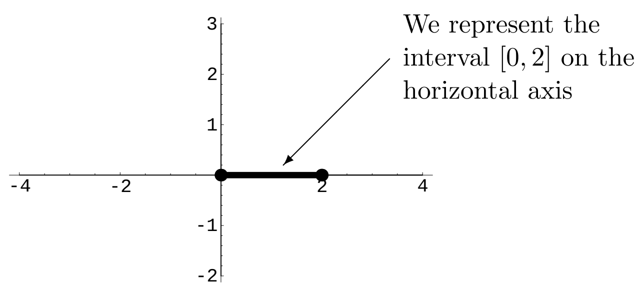
In the following example we illustrate the basic method for representing a function.

Example 6. To graphically represent the function

$$\begin{array}{c} f : [0,2] \to \mathbb{R} \\ f(x) = x^2 \end{array}$$

we will follow the following steps:

a) First, we represent the domain of the function. In this case, the domain of f is the interval $[0, 2]$. As we have indicated before, the representation of functions is carried out in the real plane. Specifically, we will represent the domain on the horizontal axis also called the abscissa axis

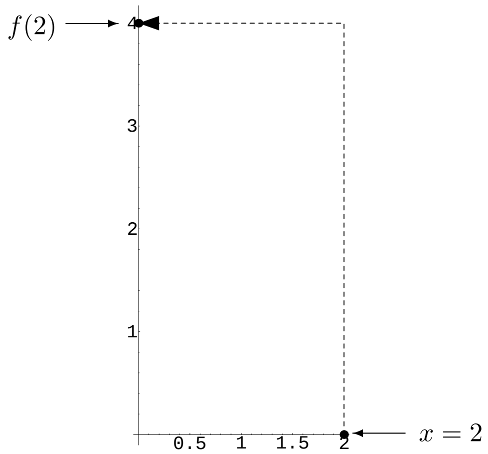


b) Next, we represent the function f by drawing its graph only over its domain, the interval $[0, 2]$, which we marked in the previous step.

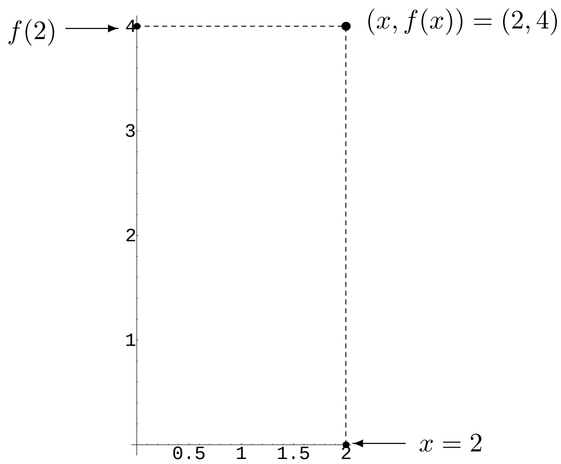
For each point x in the interval $[0, 2]$ it is easy to calculate its image $f(x)$. We will represent each point x on the horizontal axis and its image $f(x)$ on the vertical axis. Let us take, for example, $x = 2$, then

$$f(2) = 2^2 = 4,$$

and the image of $x = 2$ is, therefore, $f(2) = 4$. Graphically, we can represent this fact in the following way



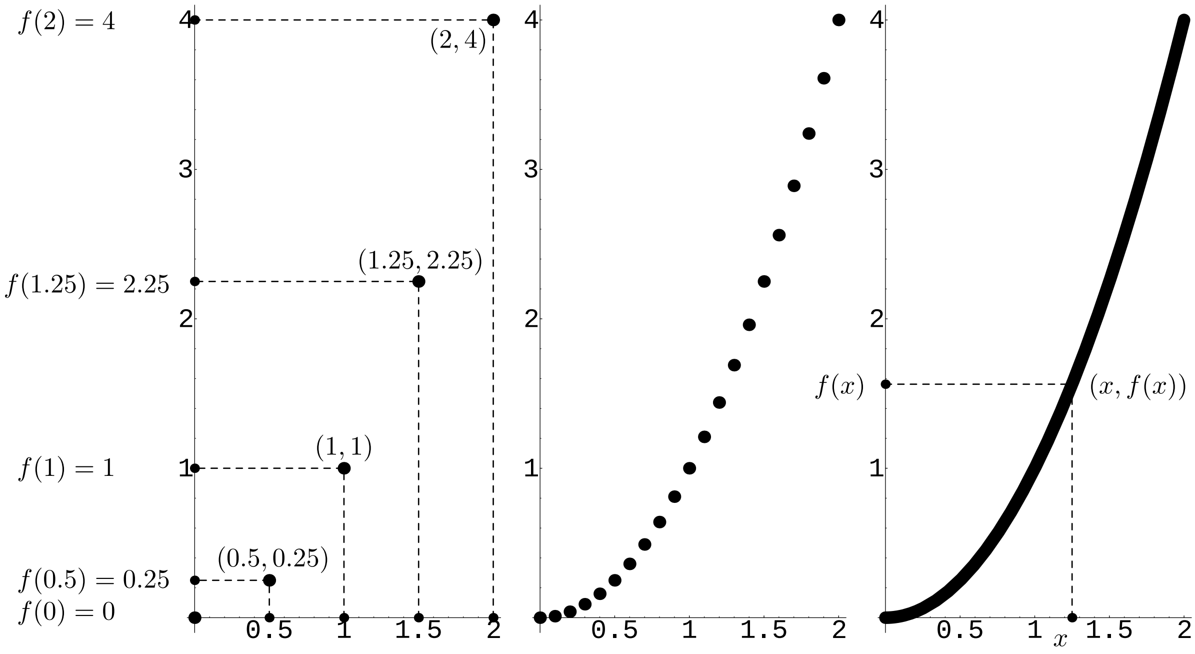
However, to draw the graph of f we will not highlight x nor $f(x)$ but the point $(x, f(x))$. In our particular example for $x = 2$ we have that $(x, f(x)) = (2, 4)$ which we represent in the following form



In this way, to obtain the graph of f we will highlight all the points of the form $(x, f(x))$ that form the set

$$\{(x, f(x)) : x \in [0, 2]\}$$

which we have called the graph set of f in the **Definition 5**. In the following three images we observe the result obtained if we repeat the process carried out before for $x = 2$ with other values such as $x = 0$, $x = 0.5$, $x = 1$, $x = 1.5$ and how, if we progressively add more points, we finally obtain the graph of f.



1.2.2 Values of a Function on an Interval

Given a function, $f : D \to \mathbb{R}$, we have seen that it is possible to compute the image and preimage of a specific element but one can also study the image and preimage of a subset.

In the previous example we present the function $P(t)$ which provides the number of individuals in month t. Once we know this function, it is possible to determine the number of individuals in a specific month. Thus, for example, in month $t = 2$ the number of inhabitants is $P(2)$ or in month $t = 4$ it is obtained by $P(4)$. However, sometimes it is also of interest to study how the population evolves, not at a single moment, but over a certain time interval. For example, we could analyze the behavior of the population between months $t = 4$ and $t = 9$, that is, in the time interval $[4, 9]$. For this we should calculate $P([4, 9])$, the image of the interval $[4, 9]$, as we will indicate in the following definition.

On other occasions, we will need to know when the population falls within a certain interval. For example, we can ask during which period the population oscillates between 3000 and 8000 inhabitants, that is, when the function P is in the interval $[3, 8]$. In this case we must calculate $P^{-1}([3, 8])$ which in the following definition we call the preimage of the interval $[3, 8]$.

Definition 7. Consider the function $f : D \to \mathbb{R}$. Then:

- Given $I \subseteq D$, the image of I under f is

$$f(I) = \{f(x) : x \in I\}.$$

- Given $J \subseteq \mathbb{R}$, the preimage of J under f is

$$f^{-1}(J) = \{x \in D : f(x) \in J\}.$$

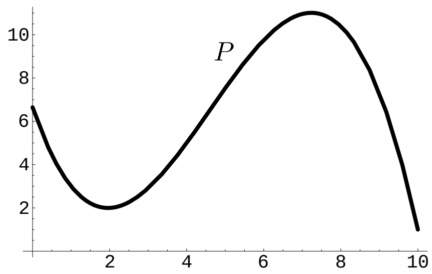
Although, in general, determining the image and preimage of a set can be a difficult computation, if we have the representation of the function, it is possible to find graphically the image and preimage of intervals in a simple way.

Examples 8.

1) The population of a certain region over the first ten months of the year is given (in thousands of individuals) by the function

$$P : [0, 10] \to \mathbb{R}$$
$$P(t) = \frac{359}{54} - \frac{1127t}{216} + \frac{731t^2}{432} - \frac{53t^3}{432}$$

whose graph is

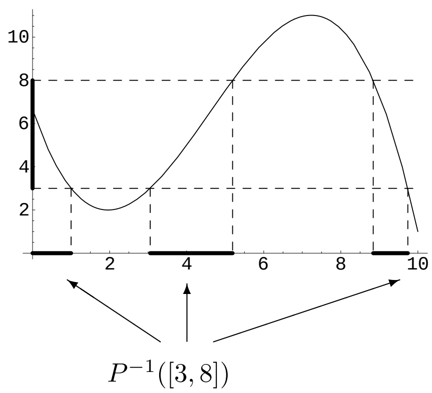


Suppose we want to determine during which months the population was between 3000 and 8000 inhabitants.

In other words, it is about determining the values of t for which $P(t)$ is between 3 and 8 (remember that P measures the population in thousands of inhabitants). That is, the preimage

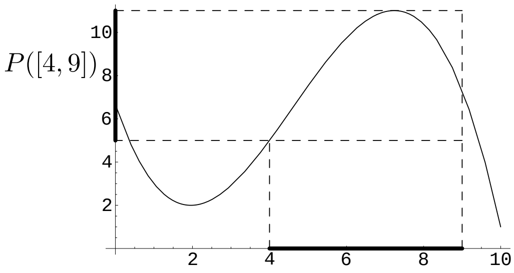
$$P^{-1}([3, 8]) = \{t \in [0, 10] : P(t) \in [3, 8]\} = \{t \in [0, 10] : 3 \leq P(t) \leq 8\}.$$

Graphically we must determine in which segments the graph of P is within the population band from 3 to 8:



2) Also study within which intervals the population moves during the fourth to ninth months.

In this case, we are interested in studying the values taken by the function on the interval $[4, 9]$, that is, we must calculate $P([4, 9])$. For this we study within which band the graph of the function moves over the segment $[4, 9]$:



1.2.3 Shape Properties of a Function

Certain characteristics of the function f determine the shape its graph will have. These are what are called 'shape properties of the function'. Some of them are compiled in the following definition.

Definition 9. A real function of a real variable, $f : D \to \mathbb{R}$, is said to be:

- *strictly increasing* if $\forall x_1, x_2 \in D$ such that $x_1 < x_2$ it holds that $f(x_1) < f(x_2)$.
- *increasing* if $\forall x_1, x_2 \in D$ such that $x_1 < x_2$ it holds that $f(x_1) \leq f(x_2)$.
- *strictly decreasing* if $\forall x_1, x_2 \in D$ such that $x_1 < x_2$ it holds that $f(x_1) > f(x_2)$.
- *decreasing* if $\forall x_1, x_2 \in D$ such that $x_1 < x_2$ it holds that $f(x_1) \geq f(x_2)$.
- *monotonic* if it is increasing or it is decreasing.
- *strictly monotonic* if it is strictly increasing or it is strictly decreasing.
- *constant* if $\exists\ a \in \mathbb{R}$ such that
$$f(x) = a,\ \forall x \in D.$$
- *injective* if $\forall x_1, x_2 \in D$ it verifies that
$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$
- *bounded above* if $\exists\ M \in \mathbb{R}$ such that $f(x) \leq M, \forall x \in D$.
- *bounded below* if $\exists\ m \in \mathbb{R}$ such that $m \leq f(x), \forall x \in D$.
- *bounded* if it is bounded above and below.

Remark. Sometimes, given a function $f : D \to \mathbb{R}$, it may happen that the entire function is not strictly increasing (resp. increasing, strictly decreasing, etc.) but that there exists some subset $\overline{D} \subseteq D$ within which it is. In such a case we will say that f is strictly increasing (resp. increasing, strictly decreasing, etc.) on $\overline{D}$.

1.3 Elementary Functions

Next we will see a list of the functions that most frequently appear in any mathematical development and we will study some of their properties and applications. These functions constitute what are called 'elementary functions'. At the end of the section we will also see how, from these elementary functions, it is possible to generate many others through composition or operation between them or through the construction of piecewise defined functions.

1.3.1 Power Functions

To define the power with base a and exponent n, a^n, we must consider several cases that we collect in the following table:

If	**then**
$n=0,\ a\neq 0$	$a^n=a^0=1$
$n\in\mathbb{R}^+,\ a=0$	$a^n=0^n=0$
$n\in\mathbb{N},\ a\in\mathbb{R}$	$a^n=a\cdot a\overset{n)}{\cdots}a\cdot a$
$n\in\mathbb{N},\ a\in\mathbb{R}-\{0\}$	$a^{-n}=\dfrac{1}{a^n}$
$n\in\mathbb{N}$ even, $a\in\mathbb{R}_0^+$	$a^{\frac{1}{n}}=z\in\mathbb{R}_0^+$ such that $z^n=a$.
$n\in\mathbb{N}$ odd , $a\in\mathbb{R}$	$a^{\frac{1}{n}}=z\in\mathbb{R}$ such that $z^n=a$.
$n=\frac{p}{q},\ a\in\mathbb{R}^+$ $p,q\in\mathbb{Z},\ q\neq 0$	$a^n=(a^p)^{\frac{1}{q}}=(a^{\frac{1}{q}})^p$.
$n\in\mathbb{R},\ a\in\mathbb{R}^+$	$\lim_{\substack{x\to n\\ x\in\mathbb{Q}}}a^x$.

We list the most important properties of powers below.

Property 10.

i) Given $a,b\in\mathbb{R}$ and $n,m\in\mathbb{N}$,

1. $0^n=0$ *and if* $a\neq 0$ *then* $a^0=1$.
2. $(ab)^n=a^nb^n$.
3. $a^{n+m}=a^na^m$.
4. $a^{nm}=(a^n)^m$.
5. $(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i}$.

ii) Given $a,b\in\mathbb{R}^+$ and $\alpha,\beta\in\mathbb{R}$

1. *If* $\alpha>0$ *then* $0^\alpha=0$.
2. *If* $\alpha<0$ *then* 0^α *is not defined.*
3. $(ab)^\alpha=a^\alpha b^\alpha$.
4. $a^{\alpha+\beta}=a^\alpha a^\beta$.
5. $a^{\alpha\beta}=(a^\alpha)^\beta$.
6. $a^{-\alpha}=\dfrac{1}{a^\alpha}$.
7. *If* $a\geq 1$ *then* $\alpha\leq\beta\Rightarrow a^\alpha\leq a^\beta$.
8. *If* $a<1$ *then* $\alpha\leq\beta\Rightarrow a^\beta\leq a^\alpha$.
9. *If* $\alpha\geq 0$ *then* $a\leq b\Rightarrow a^\alpha\leq b^\alpha$.
10. *If* $\alpha<0$ *then* $a\leq b\Rightarrow b^\alpha\leq a^\alpha$.

Once we have reviewed the concept of power, we are in a position to define what we understand by a power function.

Definition 11. We call power function with exponent $\alpha\in\mathbb{R}$ the function

$$f(x)=x^\alpha.$$

The domain and the graph of the power function depend on the exponent and can be determined taking into account what we have indicated in the definition of exponentiation. For example:

1. For $n\in\mathbb{N}$ the domain of the power function
$$f(x)=x^n$$
is $\mathbb{R}$ and therefore it will be defined for any real value of the variable x.
2. For $n\in\mathbb{N}$ the domain of the power function
$$f(x)=x^{-n}=\frac{1}{x^n}$$
is $\mathbb{R}-\{0\}$ and therefore it will be defined for any non-zero real value of the variable x.

3. For $n \in \mathbb{N}$ the domain of the power function

$$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$$

is

- $\mathbb{R}$ if n is an odd number.
- $\mathbb{R}_0^+$ if n is an even number.

4. For $n \in \mathbb{N}$ the domain of the power function

$$f(x) = x^{\frac{-1}{n}} = \frac{1}{\sqrt[n]{x}}$$

is

- $\mathbb{R} - \{0\}$ if n is an odd number.
- $\mathbb{R}^+$ if n is an even number.

5. In general for an irrational number $\alpha \in \mathbb{R} - \{0\}$ the domain of the power function

$$f(x) = x^\alpha$$

is

- $\mathbb{R}_0^+$ if $\alpha > 0$.
- $\mathbb{R}^+$ if $\alpha < 0$.

1.3.2 Exponential Functions

Definition 12. We call exponential function with base $a \in \mathbb{R}^+$ the function

$$\begin{array}{c} f : \mathbb{R} \longrightarrow \mathbb{R} \\ f(x) = a^x \end{array}.$$

Note that the domain is always the entire set $\mathbb{R}$. The exponential function that appears most frequently is the one with base the number e ($a = e$) and sometimes e^x is denoted as $\exp(x)$.

The graph of an exponential function with base a presents the following properties:

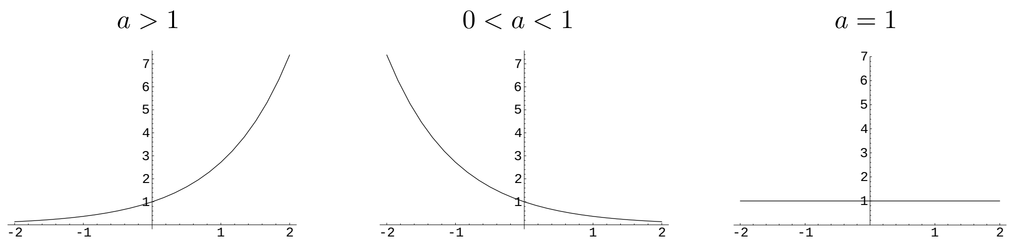
1. It always passes through the point $(0, 1)$ since

$$f(0) = a^0 = 1.$$

2. If $a = 1$ the function will be constantly equal to 1.

3. If $a > 1$ the function will be increasing.

4. If $0 < a < 1$ the function will be decreasing.



The exponential function appears in numerous financial processes. Let's see some examples of this:

- When the value in a certain period of a magnitude that varies with time is proportional to the value in the previous period, that magnitude will be adjusted by an exponential function.

Example 13. During the initial years of starting up a factory, the production increases by 10% each year. If the initial production is $P_0 = 100$ tons, we will have that:

After $t = 1$ years the production will be

$$P(1) = P_0 + 10\% \text{ of } P_0 = P_0 + 0.1P_0 = 1.1P_0.$$

After $t = 2$ years the production will be

$$P(2) = P(1) + 10\% \text{ of } P(1) = P(1) + 0.1P(1) = 1.1P(1) = 1.1^2 P_0.$$

In general, after t years the production is

$$P(t) = P(t-1) + 10\% \text{ of } P(t-1) = P(t-1) + 0.1P(t-1) = 1.1P(t-1)$$
$$= 1.1^t P_0.$$

Therefore, we can take as the production function the function

$$P(t) = P_0 1.1^t = 100 \cdot 1.1^t$$

which is an exponential function with base 1.1.

- **Compound interest.** The application of exponential-type functions in interest calculations is of special importance. Suppose we invest an initial capital P =1000€ in a bank account that pays annual interest of 8%. Then:

After $t = 1$ years, the capital in the account will be

$$P(1) = \underbrace{P}_{\text{Initial capital}} + \underbrace{8\% \text{ of } P}_{\text{Interest in the } 1^{\text{st}} \text{ year}} = P + 0.08P = (1 + 0.08)P.$$

After $t = 2$ years, the capital in the account will be

$$P(2) = \underbrace{P(1)}_{\text{Capital in account}} + \underbrace{8\% \text{ of } P(1)}_{\text{Interest of the year}} = (1 + 0.08)P + 0.08(1 + 0.08)P = (1 + 0.08)^2 P.$$

After $t = 3$ years, the capital in the account will be

$$P(3) = \underbrace{P(2)}_{\text{Capital in account}} + \underbrace{8\% \text{ of } P(2)}_{\text{Interest of the year}} = (1 + 0.08)^2 P + 0.08(1 + 0.08)^2 P = (1 + 0.08)^3 P.$$

After t years, the capital in the account will be

$$P(t) = \underbrace{P(t-1)}_{\text{Capital in account}} + \underbrace{8\% \text{ of } P(t-1)}_{\text{Interest of the year}} = (1 + 0.08)^{t-1} P + 0.08(1 + 0.08)^{t-1} P = (1 + 0.08)^t P.$$

The function $P(t)$ that provides the capital in the account in year t is therefore

$$P(t) = (1 + 0.08)^t P$$

which is an exponential function with base $a = 1 + 0.08$. For example, we have that:

- The accumulated capital in the first year will be

$$P(1) = (1 + 0.08)^1 P = 1.08 \cdot 1000 = 1080€.$$

- The accumulated capital after ten years will be

$$P(10) = (1 + 0.08)^{10} P = 1.08^{10} \cdot 1000 = 2.158 \cdot 1000 = 2158€.$$

Note that after each year the interest obtained is incorporated into the account becoming part of our capital so that for the following year that interest will in turn generate new interest.

When the interest obtained is added to the initial capital of the account, generating interest themselves for successive years, we obtain what is called 'investment at compound interest'. In the previous example there is an annual compound interest of 8%.

Usually, financial institutions tend to pay interest more than once a year. Then the nominal interest of the account is divided by the number of payments that will be made. For example if the financial institution from the previous example paid interest three times a year (i.e., every four months), the year would be divided into three periods of equal length (into three four-month periods) and the 8% annual interest would be distributed among these three periods so that the interest for each period would be $\frac{8}{3}\%$ and would be paid as follows:

After the first period (first four-month period), the capital in the account will be

$$A_1 = \underbrace{P}_{\text{Initial capital}} + \underbrace{\frac{8}{3}\% \text{ of } P}_{\text{Interest in the 1}^{\text{st}}\text{ period}} = P + \frac{0.08}{3} P = \left(1 + \frac{0.08}{3}\right) P.$$

After the second period, the capital in the account will be

$$A_2 = \underbrace{A_1}_{\text{Capital in account}} + \underbrace{\frac{8}{3}\% \text{ of } A_1}_{\text{Interest of the 2}^{\text{nd}}} = \left(1 + \frac{0.08}{3}\right) P + \frac{0.08}{3}\left(1 + \frac{0.08}{3}\right) P = \left(1 + \frac{0.08}{3}\right)^2 P.$$

After the third period, the capital in the account will be

$$A_3 = \underbrace{A_2}_{\text{Capital in account}} + \underbrace{\frac{8}{3}\% \text{ of } A_2}_{\text{Interest of the 3}^{\text{rd}}} = \left(1 + \frac{0.08}{3}\right)^2 P + \frac{0.08}{3}\left(1 + \frac{0.08}{3}\right)^2 P = \left(1 + \frac{0.08}{3}\right)^3 P.$$

In this way we see that at the end of the first year (once the three periods into which we have divided it have passed) the capital in the account will be

$$P(1) = \left(1 + \frac{0.08}{3}\right)^3 P.$$

If we want to calculate the capital in the account after two years, we must apply the same previous scheme of interest payment in three periods but taking into account that the capital at the beginning of the year is now $P(1) = \left(1 + \frac{0.08}{3}\right)^3 P$ instead of P. In such a case, it is easy to see that the capital in the second year is

$$P(2) = \left(1 + \frac{0.08}{3}\right)^3 P(1) = \left(1 + \frac{0.08}{3}\right)^3 \left(1 + \frac{0.08}{3}\right)^3 P = \left(1 + \frac{0.08}{3}\right)^{3 \cdot 2} P.$$

If we repeat this reasoning for successive years it is easy to deduce that the capital in the account after t years will be

$$P(t) = \left(1 + \frac{0.08}{3}\right)^{3t} P$$

which is again a function of exponential type. Let's calculate by means of this formula the capital in the account after one and ten years:

- The accumulated capital in the first year will be

$$P(1) = \left(1 + \frac{0.08}{3}\right)^{3\cdot 1} P = 1.026^3 \cdot 1000 = 1.082 \cdot 1000 = 1082€.$$

- The accumulated capital after ten years will be

$$P(10) = \left(1 + \frac{0.08}{3}\right)^{3\cdot 10} P = 1.026^{30} \cdot 1000 = 2.202 \cdot 1000 = 2202€.$$

We note that by dividing the interest payment into three periods, the benefits we obtain are higher. For example, we saw before that when the interest payment was made only once a year, after ten years the capital in the account was 2158€, however by dividing into three periods we have obtained 2202€.

One might wonder what would happen if we divided the year into more than three periods. For example, the interest payment could be made monthly (12 periods of one month) or daily (360 periods of one day). In general, if we have:

$$\begin{cases} \text{Initial capital} = P, \\ \text{Nominal annual interest} = r \text{ (expressed as a decimal)}, \\ \text{Number of periods into which we divide the year} = m, \end{cases}$$

the same arguments we have used before allow us to deduce that the capital in the account after t years will be

$$P(t) = \left(1 + \frac{r}{m}\right)^{mt} P. \tag{1.1}$$

This last one is the formula for the accumulated capital in an account with **compound interest with** m **compounding periods**.

- **Continuously compounded interest.**

We have seen in the previous example that if the interest payment is made in three periods, the income we will obtain is greater than wht we obtain with only one. Actually, it is easy to check using the formula for compound interest with m compounding periods that the greater the number of periods, the greater the income we obtain from interest. For example, continuing with the same example as before, after ten years we have:

- Dividing into a single period:

$$P(10) = \left(1 + \frac{0.08}{1}\right)^{1\cdot 10} P = 2.158 \cdot 1000 = 2158€.$$

- Dividing into three periods (four-monthly payments, every four months):

$$P(10) = \left(1 + \frac{0.08}{3}\right)^{3\cdot 10} P = 2.202 \cdot 1000 = 2202€.$$

- Dividing into four periods (quarterly payments, every quarter):

$$P(10) = \left(1 + \frac{0.08}{4}\right)^{4\cdot 10} P = 2.208 \cdot 1000 = 2208€.$$

- Dividing into twelve periods (monthly payments, every month):

$$P(10) = \left(1 + \frac{0.08}{12}\right)^{12\cdot 10} P = 2.219 \cdot 1000 = 2219€.$$

- Dividing into 365 periods (daily payments, every day):

$$P(10) = \left(1 + \frac{0.08}{365}\right)^{365\cdot 10} P = 2.2253 \cdot 1000 = 2225.3€.$$

We could divide the year into more periods (for example making two, three, etc. daily payments) making the number of periods m increasingly larger. When the number of periods tends to infinity the size of each period will be increasingly smaller and the interest is paid continuously at each instant. Then, in general, applying formula (1.1), the accumulated capital in the account after t years will be (we will see this better later)

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt}P = e^{rt}P$$

and the capital function we obtain is

$$P(t) = Pe^{rt}.$$

When interest is paid continuously at each instant it is said that we have an interest r compounded continuously. In the case of the previous example, if the interest is compounded continuously, after ten years the capital in the account will be:

- 8% interest compounded continuously (continuous payment of interest at each instant):

$$P(10) = Pe^{0.08t} = 2.2255 \cdot 1000 = 2225.5€.$$

In general, continuously compounded interest and interest compounded in 365 periods (compounded daily) produce very similar benefits.

1.3.3 Logarithmic Functions

Definition 14. Given $a \in \mathbb{R}^+ - \{1\}$ and $b \in \mathbb{R}^+$ we call logarithm in base a of b and denote it $\log_a b$ the number $r \in \mathbb{R}$ such that

$$a^r = b.$$

Therefore the logarithm in base a of a number b is that other number to which a must be raised to obtain b. As a consequence it is evident that

$$a^{\log_a b} = b. \tag{1.2}$$

The most usual values for the base, a, of a logarithm are:

- $a = 10$ and in that case the logarithm is called the decimal logarithm.
- $a = e$ and in that case the logarithm is called the natural or Napierian logarithm. The natural logarithm of a number $b \in \mathbb{R}^+$ is denoted by

$$\mathrm{Ln}(b), \qquad \mathrm{L}(b) \qquad \text{or simply} \qquad \log(b).$$

Properties 15. *Given $a, a' \in \mathbb{R}^+ - \{1\}$, $b, b' \in \mathbb{R}^+$ and $c \in \mathbb{R}$:*

1. $\log_a 1 = 0$.
2. $\log_a b^c = c \cdot \log_a b$.
3. $\log_a (b \cdot b') = \log_a b + \log_a b'$.
4. $\log_a \frac{b}{b'} = \log_a b - \log_a b'$.
5. $\log_a b = \dfrac{\log_{a'} b}{\log_{a'} a}$.
6. $a^c = e^{c\cdot \mathrm{Ln}(a)}$.

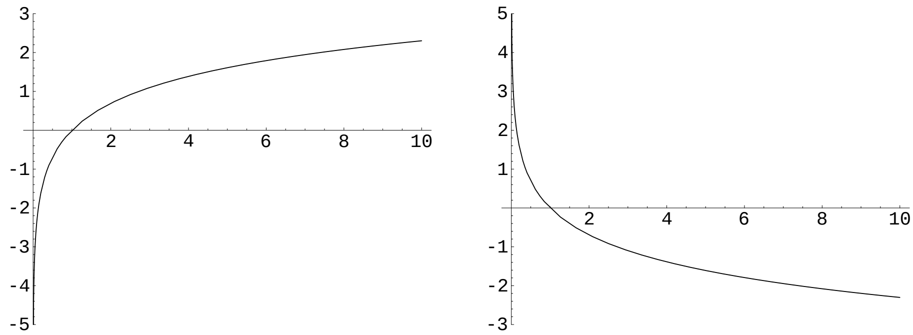
Definition 16. We call logarithmic function with base $a \in \mathbb{R}^+ - \{1\}$ the real function of a real variable

$$\begin{array}{l} f : \mathbb{R}^+ \longrightarrow \mathbb{R} \\ f(x) = \log_a x \end{array}.$$

The graph of the logarithmic function with base a presents the following properties:

1. It always passes through the point $(1, 0)$ since
$$f(1) = \log_a 1 = 0.$$
2. If $a > 1$ the function will be increasing.
3. If $0 < a < 1$ the function will be decreasing.
4. The domain of the logarithmic function is always $\mathbb{R}^+$.

$a > 1$ $\qquad\qquad$ $0 < a < 1$



As a consequence of equality (1.2), it is evident that the logarithm is the inverse function of the exponential function. We can therefore apply it in those situations where exponential functions are involved and it is necessary to solve for some of the variables.

Example 17. Suppose that in **Example 13** we want to determine the number of years that must pass for the annual production to be 200 tons. We must calculate the value of t such that

$$200 = P(t) = 100\ 1.1^t \Rightarrow 1.1^t = \frac{200}{100} = 2 \Rightarrow \underbrace{\log(1.1^t)}_{=t\log(1.1)} = \log(2) \Rightarrow t = \frac{\log(2)}{\log(1.1)} = 7.27.$$

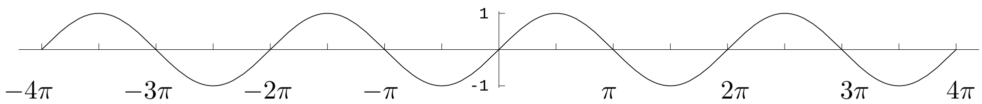
1.3.4 Trigonometric Functions

The trigonometric functions are:

The sine function. The sine function is the real function of a real variable

$$\begin{array}{c} f : \mathbb{R} \longrightarrow \mathbb{R} \\ f(x) = \text{sen}(x) \end{array},$$

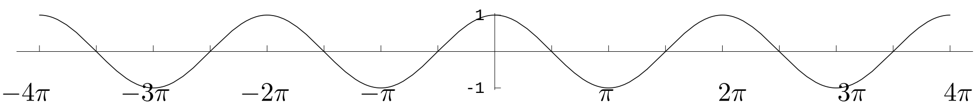
where the argument x can be interpreted as an angle measured in radians. It is a bounded function and its domain is all $\mathbb{R}$. Its graph is:



The cosine function. The cosine function is the real function of a real variable

$$\begin{array}{c} f : \mathbb{R} \longrightarrow \mathbb{R} \\ f(x) = \cos(x) \end{array},$$

where the argument x can be interpreted as an angle measured in radians. It is a bounded function, its domain is $\mathbb{R}$ and its graph is:



The tangent function. The tangent function is defined from the sine and cosine functions as follows:

$$\tan(x) = \frac{\text{sen}(x)}{\cos(x)}.$$

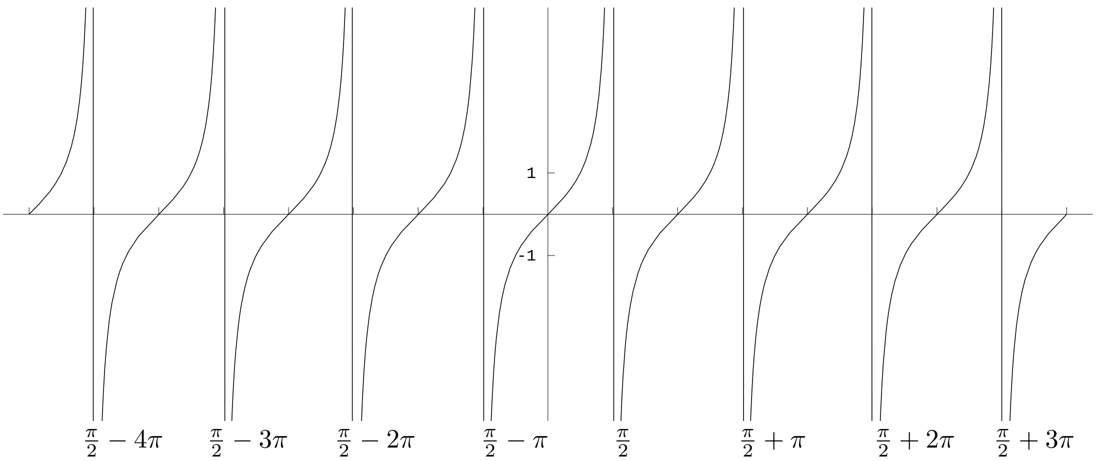
Its domain consists of those real numbers, x, for which the cosine does not vanish. Taking into account that

$$\cos(x) = 0 \Leftrightarrow x \in \{\frac{\pi}{2} + k \cdot \pi / k \in \mathbb{Z}\},$$

we will have that the domain of the tangent function is

$$\mathbb{R} - \{\frac{\pi}{2} + k \cdot \pi / k \in \mathbb{Z}\}.$$

The tangent function is an unbounded function with graph:



The secant function. The secant function is the function defined as

$$\sec(x) = \frac{1}{\cos(x)}.$$

Using the same arguments as for the tangent function we obtain that the domain of the secant function is

$$\mathbb{R} - \{\frac{\pi}{2} + k \cdot \pi / k \in \mathbb{Z}\}.$$

The secant function is also an unbounded function.

The cosecant function. The cosecant function is the function defined as

$$\text{cosec}(x) = \frac{1}{\text{sen}(x)}.$$

The cosecant function will be defined for those real values where the sine function does not vanish. Since

$$\text{sen}(x) = 0 \Leftrightarrow x \in \{k \cdot \pi / k \in \mathbb{Z}\},$$

we have that the domain of the cosecant function is

$$\mathbb{R} - \{k \cdot \pi / k \in \mathbb{Z}\}.$$

The cosecant function is an unbounded function.

We present below the basic properties of the trigonometric functions.

Properties 18. *Given* $x, y \in \mathbb{R}$

1. $\cos^2(x) + \text{sen}^2(x) = 1$.

2. $-1 \leq \cos(x) \leq 1$ *and* $-1 \leq \text{sen}(x) \leq 1$.

3. $\cos(x+y) = \cos(x)\cos(y) - sen(x)\text{sen}(y)$.

4. $\text{sen}(x+y) = \text{sen}(x)\cos(y) + \cos(x)\text{sen}(y)$.

5. *If* $x, y, x+y \in \mathbb{R} - \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}$, $\text{tg}(x+y) = \dfrac{\text{tg}(x) + \text{tg}(y)}{1 - \text{tg}(x)\text{tg}(y)}$.

6. *If* $x \in \mathbb{R} - \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}$, $1 + \text{tg}^2(x) = \sec^2(x)$.

1.3.5 Inverse Trigonometric Functions

The inverse trigonometric functions arise from the need to solve equations of the type

$$\cos(x) = K, \quad \text{sen}(x) = K \quad \text{or} \quad \tan(x) = K$$

for a known value K. Since cos, sen and tan are periodic functions, these equations will have infinitely many solutions. The inverse trigonometric functions provide us in each case with one of them from which all the others can be calculated.

Let's see below a description of the most important inverse trigonometric functions:

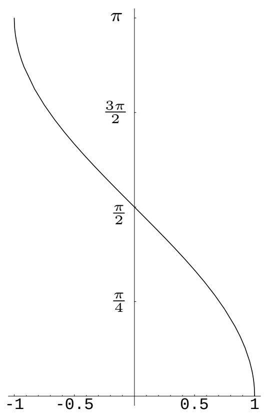
The arccos function. The arccos function is defined as

$$\begin{array}{c} \arccos : [-1, 1] \longrightarrow [0, \pi] \\ \arccos(x) = y \in [0, \pi] \text{ such that } \cos(y) = x \end{array}.$$

That is, of the infinitely many solutions that the equation

$$\cos(y) = x$$

has, $\arccos(x)$ provides us with the only one that lies between 0 and π. The arccos function is a decreasing and bounded function. Its graph is:



arccos

The arcsen function. The arcsen function is given by

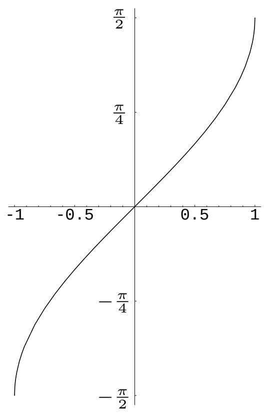
$$\begin{array}{c} \text{arcsen} : [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \text{arcsen}(x) = y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ such that } \text{sen}(y) = x \end{array} .$$

Therefore, similarly to what happened in the case of arccos, the arcsen function provides the only solution of the equation

$$\text{sen}(y) = x$$

that is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The arcsen function is an increasing and bounded function. Its graph is the following:



The arctan function. The arctg function is defined as

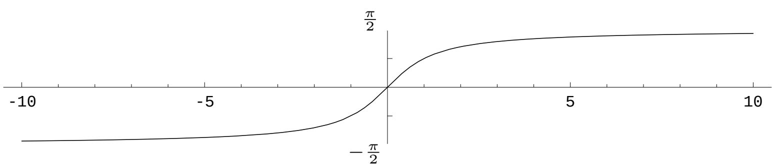
$$\begin{array}{c} \arctan : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \text{arctg}(x) = y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \text{ such that } \tan(y) = x \end{array}$$

and provides us with the only solution of the equation

$$\tan(y) = x$$

located between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The arctg function is an increasing and bounded function with the following graph:



1.3.6 Polynomial Functions

Definition 19. A polynomial function of degree $n \in \mathbb{N} \cup \{0\}$ is a function of the type

$$\begin{array}{c} f : \mathbb{R} \longrightarrow \mathbb{R} \\ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \end{array}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are called the coefficients of the polynomial function.

Note that a polynomial function of degree zero is a constant function.

A polynomial function has as its domain all of $\mathbb{R}$ and its graph can adopt various shapes depending on its degree and its coefficients. Two particular cases are of special importance:

• **Polynomials of degree 1:** A polynomial of degree one is a function of the form

$$f(x) = ax + b$$

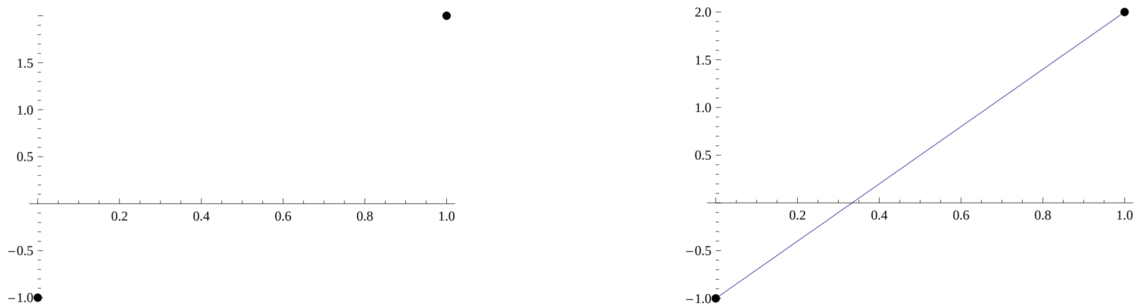
and its graph is always a straight line.

Example 20. Represent the function $f(x) = 3x - 1$.

This is a polynomial function of degree 1. Therefore, its graphical representation will be a straight line. To draw a straight line it will be sufficient to know two points it passes through. Now using the formula of $f(x)$ we know that

- At the point $x = 0$ the function takes the value $f(0) = 3 \cdot 0 - 1 = -1$.
- At the point $x = 1$ the function takes the value $f(1) = 3 \cdot 1 - 1 = 2$.

We will therefore begin by representing the value of the function at these two points and then it will be enough to draw the line joining they two to obtain the graph of $f(x)$:

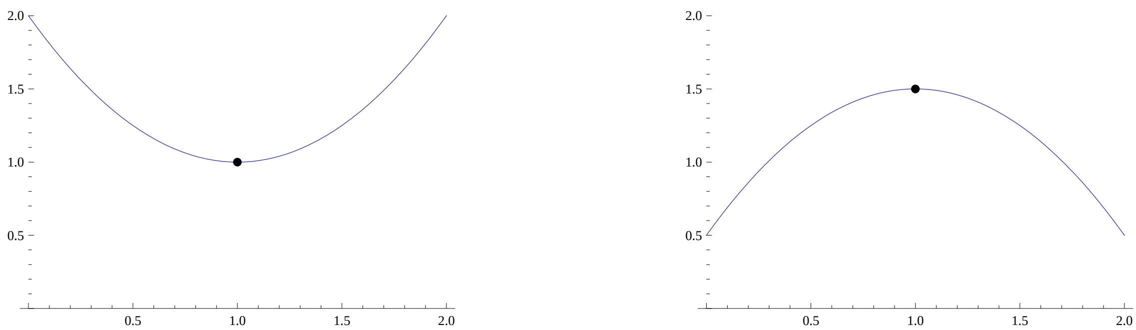


Note that, since $f(0) = -1$, the function passes through the point with coordinates (0,-1) and, since $f(1) = 2$, also through the point with coordinates $(1, 2)$.

• **Polynomials of degree 2:** A polynomial of degree two is a function of the form

$$f(x) = ax^2 + bx + c$$

and its representation is always a parabola (we will assume here that $a \neq 0$) whose graph is of the type:



As we see, a parabola is a curve in which a characteristic point appears called the vertex of the parabola, which is located at the minimum or maximum extreme of the curve and which in the previous graphs appears highlighted with a larger size. Therefore, to represent the polynomial $f(x) = ax^2 + bx + c$ we must draw a parabola and to do so, at least approximately, we must take into account the following properties:

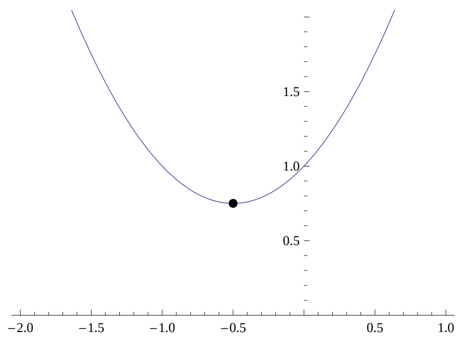
- If $a > 0$ the vertex will be below the parabola (as in the first graph) while for $a > 0$ it will be above (second graph).
- The vertex is located at the point $x = \frac{-b}{2a}$.

Example 21. Represent the function $f(x) = x^2 + x + 1$.

This is a polynomial of degree 2 and therefore is a parabola. The coefficients of the polynomial are $a = 1$, $b = 1$ and $c = 1$ (i.e., $f(x) = ax^2 + bx + c$ with $a = b = c = 1$). To represent it approximately we have that

- Since $a = 1 > 0$ the vertex will be below the parabola.
- The vertex will be located at
$$x = -\frac{b}{2a} = -\frac{1}{2}.$$
At that point the value of the function is
$$f(-\frac{1}{2}) = \left(-\frac{1}{2}\right)^2 - \frac{1}{2} + 1 = \frac{3}{4}.$$
Therefore the vertex is the point with coordinates $(-\frac{1}{2}, \frac{3}{4})$

The representation of the function will be, approximately,



Interpolation by polynomials

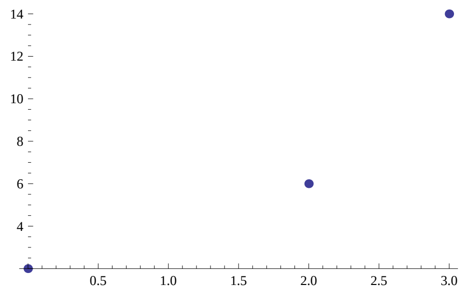
We have pointed out before that depending on the degree and the coefficients the graph of a polynomial can approximate any shape we desire. When we study an economic magnitude, many times we do not know the exact formula that describes its evolution and we will only have either its graph or isolated data from which we will have to find an appropriate formula. Polynomials are then an indispensable tool. Let's see it in the following example:

Example 22. The number of clients of a certain company varies from one year to the next. After conducting a study on the evolution of the client volume from the moment of the company's creation we have the following data:

Year	Clients (in thousands)
0	2
2	6
3	14

Note that the initial year of the study in which the company is created is year $t = 0$ and that we have information for three specific years, specifically for the initial year and the second and third subsequent years.

We wish to know the function $c(t)$ that provides the number of clients of the company in each year t. Thus, of the function we wish to find we only have the three isolated data that we see in the graph:



The question here is that we do not know the formula of the function $c(t)$ and instead we only know that $c(0) = 2$, $c(2) = 6$ and $c(3) = 14$.

We are then interested in knowing the formula of the function $c(t)$ from the data we have. Once we have that formula we will be able to predict the number of clients in other years or in future years as well as represent a graph of the function.

The problem we posed in the previous example is what is called an interpolation problem which consists of determining the formula of a function from some isolated data. It is always possible to solve this type of problem using polynomials as guaranteed by the following property:

Property 23. *Given $n+1$ distinct points $x_0, x_1, x_2, \ldots, x_n$ and the values $f_0, f_1, f_2, \ldots, f_n$, there exists a unique polynomial, $p(x)$, of degree at most n such that*

$$p(x_0) = f_0,\ p(x_1) = f_1,\ p(x_2) = f_2, \ldots,\ p(x_n) = f_n.$$

We then say that the polynomial $p(x)$ interpolates the data $f_0, f_1, f_2, \ldots, f_n$ at the points $x_0, x_1, x_2, \ldots, x_n$.

In other words, we can always solve the interpolation problem using a polynomial of degree one less than the number of data points we have. To find that polynomial there are numerous techniques and formulas such as the Lagrange or Newton interpolation formulas. We propose here a method that, in the case of a reduced number of data, is simpler. Let's see it in the following example:

Examples 24.
1) Let us compute the function $c(t)$ of the previous example.

Recall that $c(t)$ determined the number of clients in each year t for a certain company. We have three data points (the number of clients in three different years) so by applying the previous property we know that there exists a unique polynomial of degree $3 - 1 = 2$ that interpolates that data and that would be the solution to our problem. Let's calculate that polynomial.

Since it is of degree 2, it will be of the form

$$c(t) = at^2 + bt + c$$

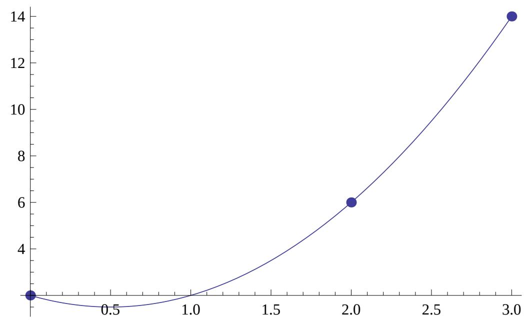
and at the same time this polynomial must satisfy $c(0) = 2$, $c(2) = 6$ and $c(3) = 14$. It will be enough to calculate the coefficients a, b and c to know this polynomial. If we use the formula and at the same time the data we know, we obtain

$$\left\{\begin{array}{rcl} 2 & = & c(0) = a \cdot 0^2 + b \cdot 0 + c, \\ 6 & = & c(2) = a \cdot 2^2 + b \cdot 2 + c, \\ 14 & = & c(3) = a \cdot 3^2 + b \cdot 3 + c. \end{array}\right. \Rightarrow \left\{\begin{array}{l} c = 2, \\ 4a + 2b + c = 6, \\ 9a + 3b + c = 14. \end{array}\right. \overset{\text{Solving the system}}{\Rightarrow} \left\{\begin{array}{l} a = 2, \\ b = -2, \\ c = 2. \end{array}\right.$$

and thus, finally we arrive at a linear system of three equations with three unknowns. Solving this system is simple using any of the usual methods. In this way, the polynomial we were looking for will be

$$c(t) = 2t^2 - 2t + 2.$$

In the following graph we can check how the graph of the polynomial $c(t)$ passes through the points corresponding to the data we had



2) Calculate the straight line that passes through the points $(2, 3)$ and $(5, -4)$.

As we have two points we know that we can solve the problem with a polynomial of degree $2 - 1 = 1$ which is, for sure, a line. Therefore we are looking for a function of the type

$$r(x) = ax + b.$$

Additionally:

- for it to pass through the point $(2, 3)$ it must hold that $r(2) = 3$,
- for it to pass through the point $(5, -4)$ it must hold that $r(5) = -4$.

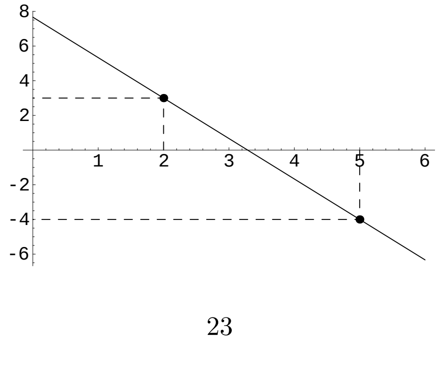
We intend to find, therefore, a polynomial that interpolates at these two points and which consequently will be of degree $2 - 1 = 1$ which fits with the fact that we want it to be a straight line. We will now apply the same technique we used in the previous example:

$$\left\{\begin{array}{rcl} 3 & = & r(2) = a \cdot 2 + b, \\ -4 & = & r(5) = a(-5) + b. \end{array}\right. \Rightarrow \left\{\begin{array}{l} 2a + b = 3, \\ -5a + b = -4. \end{array}\right. \Rightarrow \left\{\begin{array}{l} a = -\frac{7}{3}, \\ b = \frac{23}{3}. \end{array}\right.$$

Therefore,

$$r(x) = -\frac{7}{3}x + \frac{23}{3}.$$

The corresponding graph is:



1.3.7 Rational and Irrational Functions

A rational function is a function of the type

$$h(x) = \frac{f(x)}{g(x)},$$

where f and g are polynomial functions.

A rational function is therefore the quotient between two polynomials.

The domain of the rational function

$$h(x) = \frac{f(x)}{g(x)}$$

is formed by those real values for which the function g does not vanish. In general the domain of a rational function will be all $\mathbb{R}$ except a finite set of values where the polynomial g vanishes, that is, it will be of the form

$$\mathbb{R} - \{x/g(x) = 0\}.$$

The graph of a rational function can adopt various shapes just as happens in the case of polynomials. In fact we can also perform interpolation with rational functions.

Definition 25. An irrational function is a function of the form

$$h(x) = \sqrt[n]{\frac{f(x)}{g(x)}}$$

where f and g are polynomial functions and $n \in \mathbb{N}$.

An irrational function is therefore the n-th root of a rational function.

The graph of an irrational function

$$h(x) = \sqrt[n]{\frac{f(x)}{g(x)}}$$

can adopt different forms and its domain is determined as follows:

- If n is an even number the domain will be the set of numbers of $\mathbb{R}$ where the polynomial g does not vanish and where, in addition, the rational function $\frac{f(x)}{g(x)}$ is not negative, that is, it will be of the form

$$\mathbb{R} - \{x \in \mathbb{R}/g(x) = 0 \text{ or } \frac{f(x)}{g(x)} < 0\}.$$

- If n is an odd number the domain will be the set of numbers of $\mathbb{R}$ where the polynomial g does not vanish, that is, it will be of the form

$$\mathbb{R} - \{x \in \mathbb{R}/g(x) = 0\}.$$

1.4 Functions Obtained from Elementary Functions

The functions we have seen in the previous sections constitute what are called elementary functions. From them it is possible to construct new functions through two methods:

- **Through composition and operation:** It is possible to compose functions in the same way we do with mappings. Likewise, the usual operations of real numbers (addition, subtraction, multiplication, division, exponentiation) can be used to combine functions. The domain of one of these functions will be calculated taking into account the domains of the functions involved in it.
- **Through piecewise defined functions:** The elementary functions or those obtained by composition or operation of them are defined by a single formula. Piecewise functions are defined by several formulas, each acting on a different interval.

Let us study piecewise defined functions in more detail. More precisely, we give the following definition:

Definition 26. A function defined piecewise on the disjoint intervals I_1,I_2, ..., I_k, is a function of the form

$$f : I_1 \cup I_2 \cup \cdots \cup I_k \longrightarrow \mathbb{R}$$
$$f(x) = \begin{cases} f_1(x), & \text{if } x \in I_1, \\ f_2(x), & \text{if } x \in I_2, \\ \vdots & \\ f_k(x), & \text{if } x \in I_k, \end{cases}$$

where $f_1 : I_1 \to \mathbb{R}$, $f_2 : I_2 \to \mathbb{R}$,..., $f_k : I_k \to \mathbb{R}$ are known real functions of a real variable.

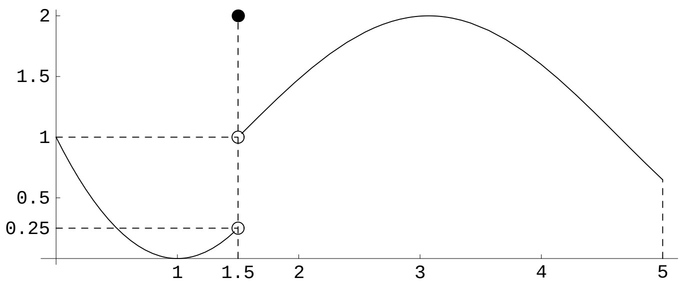
The domain of a piecewise defined function depends on the intervals on which the functions f_1, f_2,..., f_k are defined, that is to say, $I_1, I_2, \ldots, I_k$.

Examples 27.
• The function

$$f : [0, 5] \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} (x-1)^2, & \text{if } 0 \le x < 1.5, \\ 2, & \text{if } x = 1.5, \\ 1 + \operatorname{sen}(x - 1.5), & \text{if } 1.5 < x \le 5. \end{cases}$$

has graph

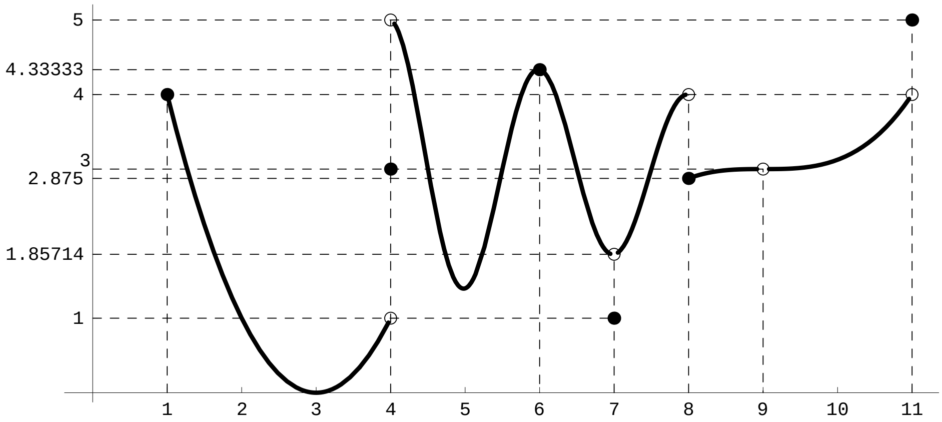


• The function

$$f : [1, 11] - \{9\} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} (x-3)^2, & \text{if } 1 \le x < 4, \\ 3, & \text{if } x = 4, \\ 3 - \frac{8\cos(\pi(x-3))}{x}, & 4 < x < 7 \\ 1, & \text{if } x = 7, \\ 3 - \frac{8\cos(\pi(x-3))}{x}, & 7 < x < 8 \\ \frac{(x-9)^3}{8} + 3, & \text{if } 8 \le x < 9 \text{ or } 9 < x \le 11, \\ 5, & \text{if } x = 11. \end{cases}$$

is a piecewise defined function with the following representation:



1.5 Limits and Continuity of a Function

Usually we represent a function by a continuous curve. However, the representation of a function is not always so regular, as gaps or jumps may appear in the graph.

Given a function $f : D \to \mathbb{R}$, at each point, $x_0 \in D$, we can consider three elements:

- The value of the function at the point, $f(x_0)$.
- If it makes sense, the value of the function to the left of the point x_0.
- If it makes sense, the value of the function to the right of the point x_0.

The idea of the value of a function to the right and left of a point corresponds to the mathematical concept of left-hand and right-hand limit of the function at that point. Although the precise mathematical definition is more complex (see the additional material on page 53), intuitively we have that:

- The value taken by the function to the left of a point x_0 is called left-hand limit, limit from below or limit from the negative part of the function at x_0 and it is denoted $f(x_0^-)$ or $\lim\limits_{x\to x_0^-} f(x)$.
- The value taken by the function to the right of a point x_0 is called the right-hand limit, limit from above or limit from the postive part of the function at x_0 and it is denoted $f(x_0^+)$ or $\lim\limits_{x\to x_0^+} f(x)$.
- The left-hand and right-hand limits are called one-sided limits. When all the one-sided limits that make sense at the point x_0 take the same value, L, we will write abbreviated $\lim\limits_{x\to x_0} f(x) = L$ and we will say that the function $f(x)$ has limit L at x_0. Conversely, when we have $\lim\limits_{x\to x_0^-} f(x) \neq \lim\limits_{x\to x_0^+} f(x)$ we say that f does not have a limit at the point x_0.

We therefore have one-sided limits, from the left and from the right, and the limit. We can study the limits at any point that the function reaches from the right or from the left. Evidently, if at a point we do not have the function to the left, it does not make sense to speak about the left-hand limit at that point and the same happens for the right-hand limit. Regarding the limit we have two possibilities:

- If both one-sided limits from the left and from the right make sense and both take the same value then we will have limit with such a value. That is,

$$\text{if } \lim_{x\to x_0^-} f(x) = L = \lim_{x\to x_0^+} f(x) \text{ then } \lim_{x\to x_0} f(x) = L.$$

- If only one of the two one-sided limits makes sense then we will not be able to compare the left-hand and right-hand ones as in the previous case and we will take as the limit that single one-sided limit that makes sense.

Moreover, we can calculate the limits at $-\infty$ and at $+\infty$ and the value of a limit can be a real number but also $-\infty$ or $+\infty$.

In the following examples we see different cases that illustrate the various possibilities we have just enumerated for limits.

Examples 28.
1) Let us consider the function from the last example and study the three mentioned elements (value of the function, value to the right and value to the left) at different points:

x_0	Value to the left $f(x_0^-) = \lim_{x \to x_0^-} f(x)$	Value at the point $= f(x_0)$	Value to the right $f(x_0^+) = \lim_{x \to x_0^+} f(x)$
$x_0 = 1$	no meaning	4	4
$x_0 = 4$	1	3	5
$x_0 = 6$	4.3333	4.3333	4.3333
$x_0 = 7$	1.85714	1	1.85714
$x_0 = 8$	4	2.875	2.875
$x_0 = 9$	3	no meaning	3
$x_0 = 11$	4	5	no meaning

It can be observed that, at each point, the three elements may make sense or only some of them. Likewise the values at each point may all be equal or different.

Intuition indicates that for the function to have regular behavior at a point x_0 the three elements must have equal values. Note that at the point $x_0 = 6$ all the values coincide and the graph of the function at that point is continuous. In contrast, at all the other points studied the values are either not all equal or some of them do not exist. This motivates that at these points the graph shows jumps or holes.

The function f is defined at all points of the interval $[1, 11]$ except the point 9 (which is why the graph shows a gap at that point). Of the infinite points at which the function is defined we have studied here only seven. Evidently the analysis we have performed for $x_0 = 1, 4, 6, 7, 8, 9, 11$ can also be performed for any other point of $[1, 11] - \{9\}$, however, the graph is continuous at all other points and therefore it is to be expected that at them the values of the function and all the limits coincide.

Regarding the limit we have the following:

1. At the point $x_0 = 4$ both the left-hand and right-hand limits make sense. Such limits reach the values:
$$\lim_{x \to 4^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 4^+} f(x) = 5.$$
Since $\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$ we conclude that the limit does not exist at the point $x_0 = 4$ ($\nexists \lim_{x \to 4} f(x)$).
2. At the point $x_0 = 9$, both one-sided limits make sense with values
$$\lim_{x \to 9^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 9^+} f(x) = 3.$$
Both coincide and therefore the function has a limit at $x_0 = 9$ whose value will be the same as that of the two one-sided limits:
$$\lim_{x \to 9} f(x) = 3.$$
Note that the function has a limit at the point $x_0 = 9$ despite the fact that the function has no value at that point (f(9) is not defined) which means that the graphics has a hole or gap at that point.

3. At the point $x_0 = 1$ only the limit from the right makes sense, whose value is

$$\lim_{x \to 1^+} f(x) = 4.$$

Since only one of the limits makes sense we cannot compare the two lateral values as in the previous cases and the limit at the point $x_0 = 1$ will simply be the value of the single one-sided limit we have. That is,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1^+} f(x) = 4.$$

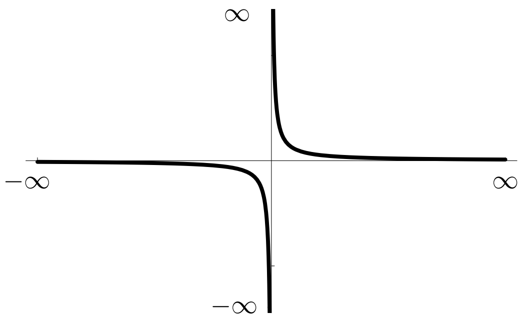
4. At the point $x_0 = 11$ we have a situation similar to the previous one since only the limit from the left makes sense:

$$\lim_{x \to 1^-} f(x) = 4.$$

Again, since only one of the one-sided limits makes sense, the limit at the point will be equal to the value of this single one-sided limit:

$$\lim_{x \to 11} f(x) = \lim_{x \to 11^-} f(x) = 4.$$

2) The domain of the function $f(x) = \frac{1}{x}$ is $D = \mathbb{R} - \{0\}$. The graph of the function is:



The one-sided limits of the function at $x_0 = 0$ are

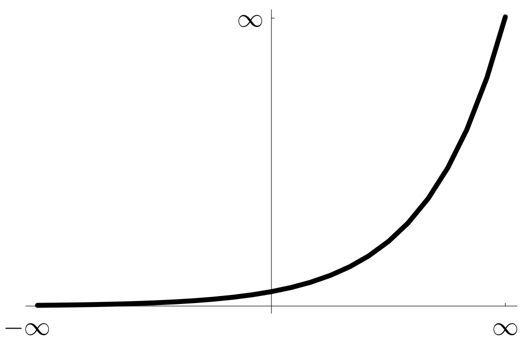
$$\lim_{x \to 0^-} f(x) = -\infty, \quad \text{and} \lim_{x \to 0^+} f(x) = \infty.$$

For the limits at $\pm\infty$ we have:

$$\lim_{x \to -\infty} f(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0.$$

Note that the values we obtain for these limits agree with the behavior of the function in the graph. Also note that this function is not defined at the point $x_0 = 0$ and despite that we can study the one-sided limits from the left and right at that point.

3) The exponential function $f(x) = e^x$ has graph



The behavior of the function is reflected in its graph but also by the value of the limits at $-\infty$ and ∞:

$$\lim_{x\to-\infty} e^x = 0 \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty.$$

As we see in these examples, the limits of a function at each point provide us with information about its graph, indicating where the function presents a continuous trace and where we will find jumps, breaks or discontinuities. The most important situations we encounter in these examples are the following:

- $\nexists f(x_0)$: The function is not defined at a point. In such a case the graph will present a gap at that point. As it happens at the point $x_0 = 9$ of **Examples 28-1)**.
- $\lim\limits_{x\to x_0^-} f(x) \neq \lim\limits_{x\to x_0^+} f(x)$: The limits from the left and from the right do not coincide. Then the graph will have a jump at the point. See the behavior of the function at the point $x_0 = 4$ of **Examples 28-1)**.
- $\lim\limits_{x\to x_0^-} f(x) = \lim\limits_{x\to x_0^+} f(x)$: The limits from the left and from the right coincide. Then the graph won't have a jump at the point. See the behavior of the function at the point $x_0 = 6$ or $x_0 = 9$ of **Examples 28-1)**.
- $\lim\limits_{x\to x_0} f(x) = f(x_0)$: The limit at the point coincides with the value of the function. In this situation the graph will have a continuous trace at that point. Then we say that the function is continuous at x_0. This is what happens at the point $x_0 = 6$ of **Examples 28-1)**. Actually, at any point different from 4, 7, 8, 9 and 11 we can check in the graph that we have no jump or gap, therefore the limit coincides with the value of the function and the function is continuous at any of the infinite points point of $[1, 11] - \{4, 7, 8, 9\}$.

There are more possibilities but these three are the most notable. In particular, the most important case is the fourth one which leads us to the definition of continuous function. It is evident that when the three elements (one-sided limits and value of the function) coincide at a point, the trace of the function at that point will be continuous.

Definition 29. Given an interval $I \subseteq \mathbb{R}$ and a function $f : I \to \mathbb{R}$, it is said that f is continuous at a point $x_0 \in I$ if we have limit at that point and the values of the function and the limit coincide. That is, if we have that

$$\lim_{x\to x_0} f(x) = f(x_0).$$

In any of the examples we have seen in this section we can check how the same function can have points where there is continuity and points where there is not. In the graph it is easy to detect at which points we have continuity since at them the trace of the function has no jumps or breaks, it is 'continuous'. In this way we verify that we have entire stretches in which the function is drawn continuously and therefore at all points of those stretches we will have continuity. This same thing, the continuity on a stretch or interval of the function, is what is reflected, with more precise language, in the following definition:

Definition 30.

- Let $f : D \to \mathbb{R}$ be a real function with domain $D \subseteq \mathbb{R}$ and let $J \subseteq D$ be a subset of D. If f is continuous at all points of J we will say that f is continuous on J.
- Let f be a real function of a real variable. If f is continuous at all points of its domain we will say that f is a continuous function.

Examples 31. Let us study again the function from the second example of **Examples 27**:

- At the point $x_0 = 6$ we have that

$$\left.\begin{array}{l}\lim\limits_{x\to 6^-} f(x) = \dfrac{13}{3} \\[2ex] \lim\limits_{x\to 6^+} f(x) = \dfrac{13}{3}\end{array}\right\} \Rightarrow \exists \lim_{x\to 6} f(x) = \frac{13}{3} = 4.\overline{3} \qquad \left.\begin{array}{r} \\ \\ f(6) = \frac{13}{3} = 4.\overline{3}\end{array}\right\} \Rightarrow \lim_{x\to 6} f(x) = f(6)$$

and the function is continuous at $x_0 = 6$. Note that continuity is equivalent to all elements of the function at that point reaching the same value, in this case $\frac{13}{3} = 4.\overline{3}$.

- At the point $x_0 = 7$,

$$\left.\begin{array}{l}\lim\limits_{x\to 7^-} f(x) = \dfrac{13}{7} \\[2ex] \lim\limits_{x\to 7^+} f(x) = \dfrac{13}{7}\end{array}\right\} \Rightarrow \exists \lim_{x\to 7} f(x) = \frac{13}{7} = 1.\overline{857142} \qquad \left.\begin{array}{r} \\ \\ f(7) = 1\end{array}\right\} \Rightarrow \lim_{x\to 7} f(x) \neq f(7)$$

In this case the one-sided limits from the left and from the right coincide and therefore the limit of the function exists at $x_0 = 7$. However, the the value of the limit does not coincide with the value of the function at the point and therefore the function is not continuous at $x_0 = 7$.

- At the point $x_0 = 9$,

$$\left.\begin{array}{l}\lim\limits_{x\to 9^-} f(x) = 3 \\[2ex] \lim\limits_{x\to 9^+} f(x) = 3\end{array}\right\} \Rightarrow \exists \lim_{x\to 9} f(x) = 3 \qquad \left.\begin{array}{r} \\ \\ f(9) \text{ is not defined}\end{array}\right\} \Rightarrow ¿\lim_{x\to 9} f(x) = f(9)?$$

Now the one-sided limits coincide and therefore the limit exists at $x_0 = 9$. However the function is not defined at $x_0 = 9$ and $f(9)$ has no value. In this situation, it does not make sense to ask whether $f(9)$ and $\lim_{x\to 9} f(x)$ have the same value or not since $f(9)$ is not defined. Therefore at $x_0 = 9$ the function f is neither continuous nor not continuous, simply it does not make sense to study continuity at that point.

- At the point $x_0 = 1$,

$$\left.\begin{array}{l}\lim\limits_{x\to 1^-} f(x) = \text{no meaning} \\[2ex] \lim\limits_{x\to 1^+} f(x) = 4\end{array}\right\} \Rightarrow \exists \lim_{x\to 1} f(x) = 4 \qquad \left.\begin{array}{r} \\ \\ f(1) = 4\end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = f(1)$$

At $x_0 = 1$ only the right-hand limit makes sense. In such a case we know that the limit will coincide with the right-hand limit and it will therefore take the value 4. Once the limit is calculated in this way it is evident that it coincides with the value of the function, $f(1) = 4$ and f is continuous at $x_0 = 1$.

- At the point $x_0 = 4$,

$$\left.\begin{array}{l}\lim\limits_{x\to 4^-} f(x) = 1 \\[2ex] \lim\limits_{x\to 4^+} f(x) = 5\end{array}\right\} \Rightarrow \nexists \lim_{x\to 4} f(x)$$

$$f(4) = 3$$

At $x_0 = 4$ the limit from the left and from the right are different and therefore the limit does not exist at this point. One of the conditions for there to be continuity at a point is that the limit exists. Therefore at $x_0 = 4$ the function is not continuous.

In summary, this is a function defined on the interval $[1, 11]$ and we have that:

⋆ The function is not defined at $x_0 = 9$. At that point the function has a gap and we can study the limits but not the continuity.

⋆ The function is not continuous at the points $x_0 = 4$, $x_0 = 7$, $x_0 = 8$ and $x_0 = 11$.

⋆ The function is continuous on the intervals $[1, 4)$, $(4, 7)$, $(7, 8)$ and $(8, 11)$. That is, at all points of those intervals.

In the previous example we see that, in general, studying continuity at a point consists of checking whether at that point the elements of the function (one-sided limits and value of the function) have the same value or not.

Remark.

- Note that while it is possible to calculate the limit of a function at a point where it is not defined, the same does not happen with continuity, since for a function to be continuous at a point it is necessary to be defined at that point.
- The continuity of a function must be studied at a point $x_0 \in \mathbb{R}$, never at $-\infty$ or $+\infty$ since the function cannot be defined at $\pm\infty$.

When a function is not continuous at a point we will say that it is discontinuous at that point. Although there are other possibilities, the two that appear in the following definition constitute the most important cases of discontinuity.

Definition 32. Let $f : D \to \mathbb{R}$ be a real function of a real variable. Then:

1. If the one-sided limits of f at $x_0 \in \mathbb{R}$ exist and are different we say that f has a jump discontinuity at the point x_0.
2. If the one-sided limits of f at $x_0 \in D$ exist and are equal to each other but different from the value of f at x_0, i.e.,
$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0),$$
then we say that f has a removable discontinuity at the point x_0.

Examples 33. Let us study again the function f from the second example of **Examples 27**. The function shows numerous points of discontinuity. For example:

1. At the point $x_0 = 4$ the function presents a jump discontinuity since the limits from the left and from the right exist but are different. So at $x_0 = 4$ we have a jump discontinuity.
2. At the point $x_0 = 7$ the function has a limit but the value of the limit does not coincide with the value of the function. At $x_0 = 7$ there is a removable discontinuity.

1.5.1 Computation of Limits and Study of Continuity

We now have a geometric idea (based on its graphical representation) of the concepts of limit and continuity. We will see that this last concept, that of continuity, is the key piece for the calculation of limits. Actually, when a function is continuous at a point, calculating its limits at that point is trivial since one simply has to substitute to calculate the value of the function at the point. Indeed, if f is continuous at the point x_0, the very definition of continuity tells us that

$$\underbrace{\lim_{x\to x_0} f(x)}_{\text{The limit at } x_0} = \underbrace{f(x_0)}_{\substack{\text{is calculated by}\\ \text{substituting } x\\ \text{by } x_0 \text{ in } f}}$$

Therefore, the essential thing now would be to know which functions are continuous since for them the calculation of limits is extremely simple. Fortunately it is possible to prove that a good part of the functions we have introduced in this chapter are continuous. This is what the following property tells us.

Theorem 34.

a) All the elementary functions introduced in Section 1.3 are continuous.

b) Every function obtained by operation (addition, subtraction, product, power, etc.) or composition of elementary functions is continuous.

Therefore elementary functions such as

$$\underbrace{f(x)=\cos(x)}_{\text{trigonometric function}} \quad , \quad \underbrace{f(x)=x^3-2x^2+4x+2}_{\text{polynomial}}, \quad \underbrace{f(x)=\log(x)}_{\text{logarithmic}}, \quad \underbrace{f(x)=\frac{1}{x}}_{\text{rational}},$$

are all continuous. If we pay attention to what we said in Definition 29, we will remember that when we say that a function is continuous, we mean that it is continuous at all points where it is defined. Let's see how we can use this to calculate limits in the following examples.

Examples 35.
1) $f(x)=\cos(x)$ is an elementary function of trigonometric type. As we saw in the corresponding section, its domain is all $\mathbb{R}$, that is, it is defined at all points. Being elementary, applying Theorem 34 we know that it is continuous at all points where it is defined and therefore it is continuous on all $\mathbb{R}$, at all points. Schematically we have

$$\left.\begin{array}{l} f(x)=\cos(x) \\ \text{elementary function} \\ \qquad + \\ f \text{ defined on } \boxed{\mathbb{R}} \end{array}\right\} \Longrightarrow f(x) \text{ continuous on } \boxed{\mathbb{R}} \quad .$$

Therefore at all points of $\mathbb{R}$ we can calculate any limit simply by substituting the value of x in the formula of $f(x)$. Thus for example

$$\lim_{x\to 3}\cos(x) = \underbrace{\cos(3)}_{\substack{\text{Substitute } x\\ \text{by } 3}} \quad , \quad \lim_{x\to 0^-}\cos(x) = \underbrace{\cos(0)}_{\substack{\text{Substitute } x\\ \text{by } 0}} = 1, \quad \lim_{x\to \pi^+}\cos(x) = \underbrace{\cos(\pi)}_{\substack{\text{Substitute } x\\ \text{by } \pi}} = -1.$$

2) $f(x)=\log(x)$ is an elementary function of logarithmic type. We know then that its domain is all $\mathbb{R}^+=(0,\infty)$, that is, it is defined for any $x>0$. Again, applying Theorem 34, being elementary, it will be continuous where it is defined and therefore on $(0,\infty)$. More briefly,

$$\left.\begin{array}{l} f(x)=\log(x) \\ \text{elementary function} \\ \qquad + \\ f \text{ defined on } \boxed{(0,\infty)} \end{array}\right\} \Longrightarrow f(x) \text{ continuous on } \boxed{(0,\infty)}.$$

In this way, at all points of $\mathbb{R}$, that is for any $x>0$, we can calculate the limits by substituting x. For example

$$\lim_{x\to 1}\log(x) = \underbrace{\log(1)}_{\text{Substitute } x \text{ by } 1} = 0, \quad \lim_{x\to 2^-}\log(x) = \underbrace{\log(2)}_{\text{Substitute } x \text{ by } 2}, \quad \lim_{x\to 10^+}\log(x) = \underbrace{\log(10)}_{\text{Substitute } x \text{ by } 10}.$$

At any of the points $x \le 0$ the function $\log(x)$ is not defined and it does not make sense to study continuity. Note that at $x=0$ we cannot study continuity but we can study the limit from the right.

3) $f(x) = 1/x$ is an elementary function of rational type. It is easy to check that it can be calculated at all points except at 0 and therefore its domain is $\mathbb{R}-\{0\}$, that is, it is defined for any $x \neq 0$. By means of Theorem 34, it will be continuous for all $x \neq 0$. That is,

$$\left.\begin{array}{l} f(x)=\dfrac{1}{x} \\ \text{elementary function} \\ \qquad + \\ f \text{ defined on } \boxed{\mathbb{R}-\{0\}} \end{array}\right\} \Longrightarrow f(x) \text{ continuous on } \boxed{\mathbb{R}-\{0\}}.$$

Therefore $f(x)$ is continuous at any point $x \neq 0$. Thus,

$$\lim_{x\to 2}\frac{1}{x} = \underbrace{\frac{1}{2}}_{\text{Substitute } x \text{ by } 2}, \quad \lim_{x\to -5^-}\frac{1}{x} = \underbrace{\frac{1}{5}}_{\text{Substitute } x \text{ by } 5}, \quad \lim_{x\to 8^+}\frac{1}{x} = \underbrace{\frac{1}{8}}_{\text{Substitute } x \text{ by } 8}.$$

At the point $x=0$ the function is not defined. Consequently it does not make sense to study continuity at that point. We can, however, calculate the limits at 0 both from above and below. However, since the function is not defined at the point we cannot calculate the limits by substituting as simply as we do in the other cases. In fact, by substituting for $x=0$ we obtain an expression that, in principle, makes no sense,

$$\lim_{x\to 0}\frac{1}{x} = \underbrace{\frac{1}{0}}_{\text{No mathematical sense}}.$$

We will soon see how to act in cases like this.

With this idea in mind let us now see how to approach the calculation of the limits of a function and the study of its continuity in different situations. There are two cases that we must study since they require different techniques:

- **Functions with a single formula**, obtained by operation or composition of elementary functions.
- **Functions with several formulas**, that is, piecewise defined functions.

Limits of Functions with a Single Formula obtained by Operation and Composition of Elementary Functions

As we have just seen, any function with a single formula obtained by composition or operation of elementary functions will always be continuous and then the basic idea for calculating a limit is simple since we only have to substitute into the formula of the function. However, when the function is not defined at the point or when we want to calculate the limits at $\pm\infty$ we will not be able to substitute so directly. In those cases something similar to what we have seen in section **3)** of **Examples 35** will happen and when we substitute we will obtain expressions that, in principle, lack clear mathematical value. In such cases we can apply the following table which, justified by precise mathematical properties and results that we don't include here, allows us to obtain the value of many of the expressions that can appear in such situations:

- $\star$ $\begin{cases} +\infty+\infty=+\infty \\ -\infty-\infty=-\infty \end{cases}$.
- $\star$ If $L\in\mathbb{R}$ then $\begin{cases} +\infty+L=+\infty \\ -\infty+L=-\infty \end{cases}$.
- $\star$ $\begin{cases} +\infty\cdot(+\infty)=+\infty \\ -\infty\cdot(-\infty)=+\infty \\ +\infty\cdot(-\infty)=-\infty \end{cases}$.
- $\star$ $\begin{cases} \text{If } L\in\mathbb{R}^+, \begin{cases} +\infty\cdot L=+\infty \\ -\infty\cdot L=-\infty \end{cases}. \\ \text{If } L\in\mathbb{R}^-, \begin{cases} +\infty\cdot L=-\infty \\ -\infty\cdot L=+\infty \end{cases}. \end{cases}$
- $\star$ If $L\in\mathbb{R}$ then $\dfrac{L}{\pm\infty}=0$.
- $\star$ $\begin{cases} \text{If } L\in(0,1) \text{ then } \begin{cases} L^{+\infty}=0 \\ L^{-\infty}=+\infty \end{cases}. \\ \text{If } L\in(1,+\infty) \text{ then } \begin{cases} L^{+\infty}=+\infty \\ L^{-\infty}=0 \end{cases}. \end{cases}$
- $\star$ $\begin{cases} \text{If } L\in\mathbb{R}^- \text{ then } (+\infty)^L=0. \\ \text{If } L\in\mathbb{R}^+ \text{ then } (+\infty)^L=+\infty. \end{cases}$
- $\star$ If $L\in\mathbb{R}$ with $L\neq 0$, then $\dfrac{L}{0}=\pm\infty^*$.

(*) The sign is determined by the rule of signs depending on the signs of L and the expression that originates the 0 in the denominator

Examples 36.

1) $\lim\limits_{x\to 0}\dfrac{1}{x^2}=\dfrac{1}{0^2}=\dfrac{+1}{+0}=+\infty.$

2) $\lim\limits_{x\to+\infty}2^{x^2+x+1}=2^{\infty^2+\infty+1}=2^{\infty}=\infty.$

3) $\lim\limits_{x\to+\infty}\left(\dfrac{1}{10}\right)^x=\left(\dfrac{1}{10}\right)^{\infty}=0.$

4) $\lim\limits_{x\to 0^+}\dfrac{1}{x}=\dfrac{+1}{+0}=+\infty.$

5) $\lim\limits_{x\to 0^-}\dfrac{1}{x}=\dfrac{+1}{-0}=-\infty.$

The expressions that appear in the previous table admit a unique and direct resolution by applying the different points collected in it. However, in the calculation of limits, once the substitution has been performed, we can encounter another type of expressions whose value is indeterminate and depends on the specific functions we are handling, and general rules cannot be established for them. This type of expressions are what are called indeterminate forms. The most important ones are those that appear in the following list:

$$\infty-\infty,\quad \frac{0}{0},\quad \frac{\infty}{\infty},\quad \infty\cdot 0,\quad 1^{\infty},\quad 0^0,\quad 0^{\infty},\quad \infty^0.$$

The computation of the value of a limit in which an indeterminate form appears is, in general, a complex problem and only in some cases we will we be able to solve it in a simple way. We give below a list of rules that allow to solve the simplest cases:

i) To calculate the limit at $\pm\infty$ of a polynomial function,

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n,$$

we will factor out x^n which will yield

$$f(x) = x^n \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x} + a_n \right)$$

and then we calculate the limit.

ii) If f is a rational function of the form $f(x) = \dfrac{f_1(x)}{f_2(x)}$ then:

ii.a) If $x_0 = +\infty, -\infty$, factor out the highest power of x in the numerator and denominator and subsequently simplify.

ii.b) If $x_0 = 0$ simplify between numerator and denominator.

iii) If the indetermination $\infty - \infty$ appears in expressions of the type

$$\sqrt{f_1(x)} - \sqrt{f_2(x)}$$

we will multiply and divide by the conjugate $\sqrt{f_1(x)} + \sqrt{f_2(x)}$. Note that by multiplying the expression by its conjugate we obtain the following result

$$(\sqrt{f_1(x)} - \sqrt{f_2(x)})(\sqrt{f_1(x)} + \sqrt{f_2(x)}) = f_1(x) - f_2(x).$$

This type of technique can also be used when expressions like:

$$f_1(x) - \sqrt{f_2(x)} \quad \text{or} \quad \sqrt{f_1(x)} - f_2(x).$$

appear.

iv) If we obtain an indetermination of the type 1^∞ we will apply the property:

Property 37. *Let f be a positive real function and g a real function. Then*

$$\lim_{x \to x_0} f(x)^{g(x)} = e^L \Leftrightarrow \lim_{x \to x_0} g(x)(f(x) - 1) = L$$

where L can be a real number, $+\infty$ or $-\infty$ and instead of x_0 we can put x_0^+, x_0^-, $+\infty$ or $-\infty$.

v) For an indetermination of the type 0^0, 0^∞ or ∞^0, we can use the following transformation

$$f(x)^{g(x)} = e^{g(x)\log(f(x))}$$

or we can take into account that

$$\lim_{x \to 0} x^x = 1 \qquad \text{and} \qquad \lim_{x \to +\infty} x^{\frac{1}{x}} = 1.$$

vi) Given $a \in \mathbb{R}^+$ it holds that

$$\lim_{x \to +\infty} \frac{\log(x)}{x^a} = \lim_{x \to +\infty} \frac{x^a}{e^x} = \lim_{x \to +\infty} \frac{e^x}{x^x} = 0.$$

This chain of equalities is known as the scale of infinities.

vii) Note that the functions cosine, sine, tangent, secant and cosecant are periodic functions that have no limit at $+\infty$ nor at $-\infty$. Also note that for $k \in \mathbb{Z}$

$$\lim_{x\to(\frac{\pi}{2}+k\pi)^+} \tan(x) = -\infty \quad \textit{and} \quad \lim_{x\to(\frac{\pi}{2}+k\pi)^-} \tan(x) = +\infty,$$

as well as

$$\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2} \quad \textit{and} \quad \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}.$$

viii) For the logarithm function we have that if $a > 1$,

$$\lim_{x\to+\infty} \log_a(x) = +\infty \text{ and } \lim_{x\to 0} \log_a(x) = -\infty$$

and if $a < 1$ then

$$\lim_{x\to+\infty} \log_a(x) = -\infty \text{ and } \lim_{x\to 0} \log_a(x) = +\infty.$$

Piecewise Defined Functions

Given a piecewise function defined on the intervals I_1, I_2, ...,I_k we can study the limits, one-sided limits or the continuity. For this we follow the following indications:

1. Within each of the intervals I_1, I_2,..., I_k the function will be defined by a single formula and then we will apply the methods indicated in the previous subsection.
2. At the boundary points of the intervals I_1, I_2,...,I_k we will calculate the one-sided limits taking into account in each case how the function is defined and then:
   - if the one-sided limits coincide, the function will have a limit.
   - if the function has a limit and this coincides with the value of the function then the function will be continuous.

Example 38. Let us study the continuity on $\mathbb{R}$ of the function from **Example 27** whose definition we recall below:

$$f : [1, 11] - \{9\} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} (x-3)^2, & \text{if } 1 \le x < 4, \\ 3, & \text{if } x = 4, \\ 3 - \frac{8\cos(\pi(x-3))}{x}, & \text{if } 4 < x < 7, \\ 1, & \text{if } x = 7, \\ 3 - \frac{8\cos(\pi(x-3))}{x}, & \text{if } 7 < x < 8, \\ \frac{(x-9)^3}{8}, & \text{if } 8 \le x < 9 \text{ or } 9 < x < 11, \\ 5, & \text{if } x = 11. \end{cases}$$

We observe that it is a piecewise function defined on several intervals and that it has points of definition change at $x_0 = 1, 4, 7, 8, 9$ and 11. To study the continuity we will begin with the intervals where the function has a single formula and leave the points of definition change for the end. In this way we solve the problem in two phases:

First phase: Within each of the open intervals $I_1 = (1, 4)$, $I_2 = (4, 7)$, $I_3 = (7, 8)$, $I_4 = (8, 9)$, $I_5 = (9, 11)$ the function is defined by a single formula (in I_1 by $(x-3)^2$, in I_2 by $3 - \frac{8\cos(\pi(x-3))}{x}$, ...), which is obtained by operation (addition, product,...) and composition of continuous functions (elementary functions), whereby the function f is continuous on each of those intervals.

Second phase: Let us now study the points of definition change. For each of them we must analyze the three significant elements of the function at that point, that is, left-hand and right-hand limits and value of the function (all of them have already been calculated for this function in **Example 31**):

- At point $x_0 = 1$, we saw that

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^+} f(x) = 4 = f(1),$$

  so that f is continuous at $x_0 = 1$.

- At the point $x_0 = 4$ we had that

$$\nexists \lim_{x\to 4} f(x),$$

  so that f is not continuous at $x_0 = 4$.

- At the point $x_0 = 7$,

$$\exists \lim_{x\to 7} f(x) = \frac{13}{7} \neq 1 = f(7),$$

  so that f is not continuous at $x_0 = 7$.

- At the point $x_0 = 8$,

$$\lim_{x\to 8^-} f(x) = 4 \neq -\frac{1}{8} = \lim_{x\to 8^+} f(x),$$

  so that $\nexists \lim_{x\to 8} f(x)$ and, consequently, f is not continuous at $x_0 = 8$.

- For $x_0 = 11$ we have that

$$\lim_{x\to 11} f(x) = \lim_{x\to 11^-} f(x) = 1 \neq 5 = f(11),$$

  so that f is not continuous at $x_0 = 11$.

- Finally, at $x_0 = 9$ the function f is not defined, so at those points it does not make sense to study the continuity of f.

In summary:

I. The function f is continuous on the intervals $(1, 4)$, $(4, 7)$, $(7, 8)$, $(8, 9)$ and $(9, 11)$ and at the point $x_0 = 1$. Therefore, f is continuous on

$$[1, 4) \cup (4, 7) \cup (7, 8) \cup (8, 9) \cup (9, 11).$$

II. The function f is not continuous at the points $x_0 = 4, 7, 8$ and 11. At the point $x_0 = 9$ the function is not defined and therefore it does not make sense to study its continuity.

III. At the remaining points, before $x_0 = 1$ and after $x_0 = 11$, the function f is not defined and it is not possible to study its continuity.

1.5.2 Limits at ∞ and Future Tendency

The limit of a function at $+\infty$ has important interpretations in the study of the future trend or tendency of processes represented by a function. When the function $f(t)$ provides the value of a certain magnitude that varies with respect to time, the limit

$$\lim_{t\to\infty} f(t)$$

can be interpreted as the future tendency in the evolution of that magnitude.

Example 39. The annual maintenance expenses of a factory are given by the function

$$f(t) = \frac{6t^2 - t + 5}{t^2 + 1}$$

which provides the expenses in millions of euros in the year t. The tendency of the expenses for successive years (after a sufficiently large number of years) is given by the limit

$$\lim_{t\to\infty} \frac{6t^2 - t + 5}{t^2 + 1}.$$

This is the limit of a rational function when t tends to ∞ in which an indetermination appears. As we have seen on page 35, to calculate the limit we factor out the highest power of t in the numerator and denominator:

$$\lim_{t\to\infty} \frac{6t^2 - t + 5}{t^2 + 1} = \lim_{t\to\infty} \frac{t^2\left(6 - \frac{1}{t} + \frac{5}{t^2}\right)}{t^2\left(1 + \frac{1}{t^2}\right)} = \lim_{t\to\infty} \frac{6 - \frac{1}{t} + \frac{5}{t^2}}{1 + \frac{1}{t^2}} = \frac{6 - \frac{1}{\infty} + \frac{5}{\infty^2}}{1 + \frac{1}{\infty^2}} = \frac{6 - 0 + 0}{1 + 0} = 6.$$

When a large number of years have passed, the expense in the factory tends to approach 6 million euros. This figure marks the trend of the expense for the future.

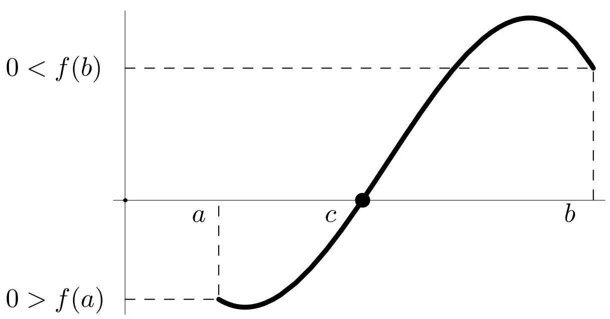
1.5.3 Bolzano's Theorem. Bisection Method. Solving Inequalities

Bolzano's theorem, although it a simple theorem, has multiple applications in the handling of continuous functions. Let's see its formulation along with some important applications of the result.

Sometimes it is of interest to determine for which values of x, a certain function $f(x)$ vanishes. In other words, we are interested in solving the equation

$$f(x) = 0.$$

Bolzano's theorem sets the simplest conditions under which we can at least affirm that this value x exists. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < 0$ and $f(b) > 0$. Continuity prevents the function from having jumps or gaps so with this hypothesis, necessarily, the graph of f must be something like



From the graph it is evident that the function f must cut the axis $y = 0$ at some point. In the graph, that point has been denoted as c. In essence this is what Bolzano's Theorem, which we state below, asserts.

Property 40 (Bolzano's Theorem)**.** *Let f be a real function defined on the interval $[a, b]$, continuous on this interval and such that $f(a) \neq 0$, $f(b) \neq 0$. Then, if it is verified that the sign of $f(a)$ is different from the sign of $f(b)$, there exists $c \in (a, b)$ such that $f(c) = 0$.*

Two points must be taken into account about this theorem:

- Bolzano's theorem guarantees the existence of c but does not offer any method to determine its exact value.
- Bolzano's theorem affirms that there exists a value at which the function f vanishes but does not prevent f from vanishing at other different points.

Certainly, Bolzano's Theorem by itself does not allow the calculation of the point c at which f vanishes. However, its repeated application following an adequate scheme makes it possible to at least approximate the value of c. Let's see next that scheme which is called the **bisection method**.

The bisection method

Suppose we have a function $f : [a, b] \to \mathbb{R}$ under the conditions of Bolzano's theorem and therefore

$$f(a)f(b) < 0 \quad \text{(i.e., } f(a) \text{ and } f(b) \text{ have different signs).}$$

Then there exists $c \in (a, b)$ such that $f(c) = 0$. We will assume that by some means we know that c is the only solution of that equation in the interval (a, b). To approximate c we will use the following iterative method:

Step 0: We call $a_0 = a$ and $b_0 = b$.

Step 1: Calculate the midpoint between a_0 and b_0,

$$c_0 = \frac{a_0 + b_0}{2}.$$

We have three possibilities of which necessarily one will occur:

- If $f(c_0) = 0$ we will have finished since we have found the solution which would be $c = c_0$.
- If $f(c_0)f(a_0) < 0$ we take $a_1 = a_0$ and $b_1 = c_0$.
- If $f(c_0)f(b_0) < 0$ we take $a_1 = c_0$ and $b_1 = b_0$.

In the last two cases we will not have calculated the solution but by applying Bolzano's Theorem we know that $c \in (a_1, b_1)$. The advantage is that the length of interval (a_1, b_1) is half of the one of (a_0, b_0) and we have located c in a smaller stretch.

Step $\mathbf{k+1}$: Assuming that in step k we have calculated a_k and b_k, compute the midpoint of both,

$$c_k = \frac{a_k + b_k}{2}.$$

We again have the following three possibilities:

- If $f(c_k) = 0$ we will have finished since we have found the solution which would be $c = c_k$.
- If $f(c_k)f(a_k) < 0$ we take $a_{k+1} = a_k$ and $b_{k+1} = c_k$.
- If $f(c_k)f(b_k) < 0$ we take $a_{k+1} = c_k$ and $b_{k+1} = b_k$.

Again Bolzano's Theorem guarantees that $c \in (a_{k+1}, b_{k+1})$.

In each step of the method the exact solution, c, belongs to an increasingly smaller interval. Each of the calculated values, $c_0, c_1, \ldots, c_k$ are approximations of c. It is clear that the error committed if we take as approximation the value c_k obtained in step k is

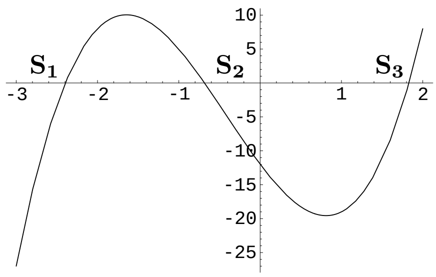
$$E_k = |c - c_k| \le \frac{b_k - a_k}{2} = \frac{b_0 - a_0}{2^{k+1}}.$$

If we fix a tolerance or maximum error E_M, for the error to be less than the fixed one we need that

$$E_k = \frac{b_0 - a_0}{2^{k+1}} \leq E_M \Rightarrow \frac{\log\left(\frac{b_0-a_0}{E_M}\right)}{\log(2)} \leq k+1 \Rightarrow \frac{\log\left(\frac{b_0-a_0}{E_M}\right)}{\log(2)} - 1 \leq k$$

and thus we can estimate the number of steps necessary for the error to be the desired one.

Example 41. Let's calculate the solutions of $4x^3 + 5x^2 - 16x = 12$. For this let's take the function $f(x) = 4x^3 + 5x^2 - 16x - 12$ (which is a polynomial and therefore continuous) and apply the bisection method to determine the values of x for which $f(x) = 0$. If we observe the representation of the function,



we will notice that the equation $f(x) = 0$ has three solutions **S1**, **S2** and **S3**. For each of them we must apply the method separately. Therefore we apply three times taking the initial data a_0 and b_0 as we indicate below:

- It is observed in the graph that **S1** is between -3 and -2 so we will take $a_0 = -3$, $b_0 = -2$.
- The second solution, **S2**, is between -1 and 0, so in this case we will choose $a_0 = -1$, $b_0 = 0$.
- For the third solution, **S3**, it is clear that an adequate choice is $a_0 = 1$, $b_0 = 2$.

Applying the bisection algorithm in these three cases we obtain the following data:

	S1		**S2**		**S3**	
n	a_k	b_k	a_k	b_k	a_k	b_k
0	-3	-2	-1	0	1	2
1	-2.5	-2	-1	-0.5	1.5	2
2	-2.5	-2.25	-0.75	-0.5	1.75	2
3	-2.5	-2.375	-0.75	-0.625	1.75	1.875
4	-2.4375	-2.375	-0.6875	-0.625	1.8125	1.875
5	-2.40625	-2.375	-0.6875	-0.65625	1.8125	1.84375
6	-2.40625	-2.39063	-0.6875	-0.671875	1.82813	1.84375
7	-2.39844	-2.39063	-0.6875	-0.679688	1.82813	1.83594
8	-2.39844	-2.39453	-0.6875	-0.683594	1.82813	1.83203
8	-2.39844	-2.39648	-0.685547	-0.683594	1.83008	1.83203
10	-2.39746	-2.39648	-0.68457	-0.683594	1.83008	1.83105

From these results we can give the following approximations

$$\mathbf{S1} \approx c_{10} = \frac{a_{10} + b_{10}}{2} = -2.39697, \quad \mathbf{S2} \approx c_{10} = \frac{a_{10} + b_{10}}{2} = -0.684082, \quad \mathbf{S3} \approx c_{10} = \frac{a_{10} + b_{10}}{2} = 1.83057.$$

In the three cases, since the length of the initial interval is always equal to one, the estimation of the committed error is

$$E_{10} = \frac{1}{2^{11}} = 0.000488281.$$

Solving Inequalities

We often need to determine for which values an inequality holds, that is, we need to 'solve the inequality'. In what follows we will see that if we are capable of solving the equality corresponding to the inequality we are working with, we will be able to solve the inequality directly.

Suppose we want to solve the inequality

$$f(x) > 0$$

where $f(x)$ is a certain known formula or function that we know is continuous. We will follow for this the following steps:

1) We solve the equality corresponding to the inequality in question, that is, the equality

$$f(x) = 0.$$

Example 42. Let's solve the inequality

$$x^3 - 9x^2 + 23x - 12 > 3.$$

For this (the function that appears in the inequality is continuous because it is a polynomial) we begin by first solving the equality

$$x^3 - 9x^2 + 23x - 12 = 3.$$

We will subtract the 3 so that we will solve the following equation which is equivalent to the previous one:

$$x^3 - 9x^2 + 23x - 15 = 0.$$

For this equation we can apply Ruffini's method as follows,

	1	−9	23	−15
1		1	−8	15
	1	−8	15	0
3		3	−15	
	1	−5	0	
5		5		
	1	0		

Therefore, the solutions of the equation will be

$$x = 1, \quad x = 3 \quad x = 5.$$

Note: Actually, in this example we do not have an inequality of the type $f(x) > 0$, however it is evident, as we have seen before, that it would suffice to subtract the 3 to leave our inequality in the form

$$x^3 - 9x^2 + 23x - 12 - 3 > 0$$

which does fit the format $f(x) > 0$.

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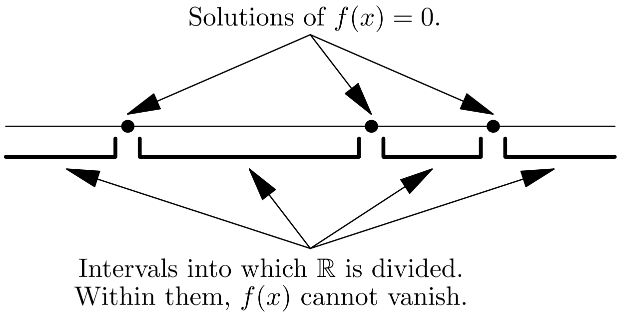
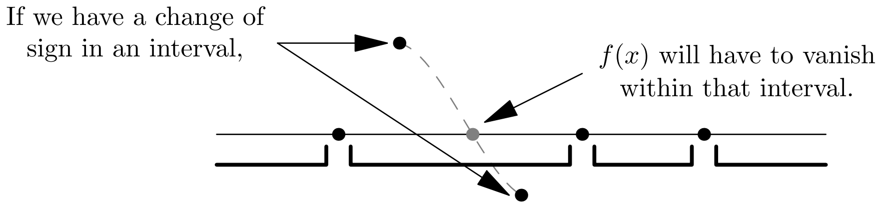
2) The solutions of the equality posed in the previous step will divide the real line, $\mathbb{R}$, into several intervals within which the function $f(x)$ can no longer vanish.



If we know that $f(x)$ is continuous, within each of those intervals we cannot have any change of sign since that would imply, applying Bolzano's Theorem, that $f(x)$ vanished somewhere in the interior of the interval (i.e., we would have solutions of $f(x) = 0$ different from those we have calculated) which is impossible.

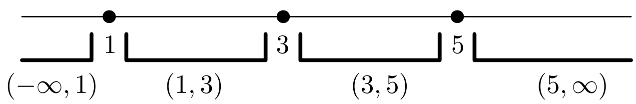


Therefore, the sign within each interval is always the same or, said in another way, in each interval we will have that, either $f(x) > 0$, or $f(x) < 0$. Then, it will suffice to check one point from each interval to detect in which of them we have $f(x) > 0$. The intervals that we obtain in this way for $f(x) > 0$, will constitute the solution of the inequality.

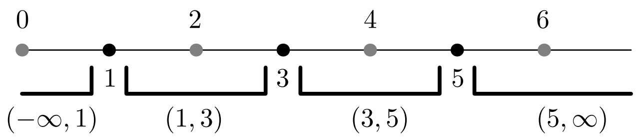
Example 43. The solutions of the equation

$$x^3 - 9x^2 + 23x - 15 = 0$$

that we calculated before ($x = 1, 3, 5$) divide the real line into four intervals,



We know that within these four intervals, there cannot be changes of sign so it will be sufficient to check what happens at any point of each one of them to deduce what the behavior will be in the whole interval. Let's take then, a point inside each of the intervals



and let's check if the inequality holds for these points:

- At the point $0 \in (-\infty, 1)$, we have that

$$\underbrace{0^3 - 9 \cdot 0^2 + 23 \cdot 0 - 12}_{=-12} \not> 3.$$

Therefore, at the point $x = 0$ the inequality does not hold and consequently

$$x^3 - 9x^2 + 23x - 12 > 3 \text{ will not hold for any point of the interval } (-\infty, 2).$$

- At the point $2 \in (1,3)$, we have that

$$\underbrace{2^3 - 9 \cdot 2^2 + 23 \cdot 2 - 12}_{=6} > 3.$$

Therefore, at the point $x = 2$ the inequality does hold and consequently

$$x^3 - 9x^2 + 23x - 12 > 3 \text{ will be true for all points of the interval } (1,3).$$

- At the point $4 \in (3,5)$, we have that

$$\underbrace{4^3 - 9 \cdot 4^2 + 23 \cdot 4 - 12}_{=0} \not> 3.$$

Therefore, at the point $x = 4$ the inequality does not hold and consequently

$$x^3 - 9x^2 + 23x - 12 > 3 \text{ will not be true for any point of the interval } (3,5).$$

- Finally, at the point $6 \in (5,\infty)$, we have that

$$\underbrace{6^3 - 9 \cdot 6^2 + 23 \cdot 6 - 12}_{=18} > 3.$$

Therefore, at the point $x = 6$ the inequality holds and

$$x^3 - 9x^2 + 23x - 12 > 3 \text{ will be true in the whole interval } (5,\infty).$$

In conclusion, we deduce that the solutions of the initially posed inequality are the points of the intervals $(1,3)$ and $(5,\infty)$ or, what is the same, the set of solutions is

$$(1,3) \cup (5,\infty).$$

1.6 Additional Material

1.6.1 Neighborhoods of a Point

Extension of concepts about intervals. Page 3

Frequently, it is not possible to handle exact data; instead, intervals are used that adjust around the exact value which we do not have. For example, it is difficult to know with total precision the number of meters of cable needed to surround a certain property; however, it is more reasonable to know an interval that contains this quantity. Thus, generally we will not have data of the type

'836.0012345601 meters of cable are needed'

but rather of the form

'between 830 and 840 meters of cable are needed'.

The mathematical concept that corresponds to the notion of an interval that encompasses a certain exact number is that of a neighborhood of a point.

Definition 44.

- Given $x_0 \in \mathbb{R}$, we call a neighborhood of x_0 any open interval, $I = (a,b)$, such that $x_0 \in I$ ($\Leftrightarrow a < x_0 < b$).

- We call a neighborhood of $+\infty$ any open interval of the form $(a, +\infty)$, with $a \in \mathbb{R}$.

- We call a neighborhood of $-\infty$ any open interval of the form $(-\infty, a)$, with $a \in \mathbb{R}$.

It is clear that for the same point we can find a multitude of different neighborhoods. It also seems clear that the smaller the neighborhood of a point, the more accurately it will approximate that point.

Examples 45.
1) The intervals $(-1, 5)$, $(2, 1000)$ or $(3.9, 4.1)$ are all neighborhoods of the point 4 since this point belongs to each of them.

2) The interval $(-3, +\infty)$ is a neighborhood of $+\infty$.

3) The interval $(-\infty, 2)$ is a neighborhood of $-\infty$.

4) The amount of waste produced by a factory is exactly 1345.342 kg daily. This quantity can be approximated by giving either of the neighborhoods $(1000, 1500)$ or $(1300, 1350)$. The second of these neighborhoods is much smaller (since its length is equal to $1350 - 1300 = 50$ which is less than $1500 - 1000 = 500$, the length of the first one) and therefore approximates the exact quantity with greater precision.

1.6.2 Composition and Inverse of Functions

Extension of concepts about functions. Page 9

When working with mappings and functions, the concepts of composition and inverse are fundamental. Apart from the mathematical definition of both concepts, we will see here how both have a direct interpretation when we work with functions that represent phenomena and processes from reality.

Composition of Functions

One basic concept in functions analysis is the composition.

Definition 46. Given two functions $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $g : f(I) \subseteq \mathbb{R} \to \mathbb{R}$, the composition of g and f, denoted $g \circ f$, is defined by

$$g \circ f : I \subseteq \mathbb{R} \to \mathbb{R}$$
$$(g \circ f)(x) = g(f(x)).$$

That is to say, the composition of g and f is the new function that we obtain if we first apply f and later g. This basic mathematical concept has an important interpretation as a change of variable. Let us illustrate this with various examples.

Suppose we are working with a certain magnitude M that varies as a function of the variable x according to the function $M(x)$.

Example 47. If we study a company that supplies fuel for heating systems, we could take

$$M = \text{number of employees of the company}$$
$$x = \text{number of clients of the company}$$

so that the number of employees hired depends on the number of clients that the company has to serve according to the function

$$M(x) = \text{employees needed to serve } x \text{ clients.}$$

Suppose additionally that the variable x depends on another variable t according to the function $x(t)$.

Example 48. Continuing with the previous example, the number of clients could depend on the average temperature each month, so we would be handling the variables

$$x = \text{number of clients of the company}$$
$$t = \text{average monthly temperature}$$

related by the function

$$x(t) = \text{number of clients when the average monthly temperature is } t.$$

Once the two functions $M(x)$ and $x(t)$ are known, we could ask if it is possible to obtain M as a function of the variable t (instead of as a function of x). It is evident that knowing the value of t, the value of x will be given by the function $x(t)$ and then the value of M will be $M(x(t))$. Schematically we have

Value of t		**Value of x**		**Value of M**
t	$\Rightarrow$	$x(t)$	$\Rightarrow$	$M(x(t))$

Now, $M(x(t)) = (M \circ x)(t)$ is the composition of the functions $M(x)$ and $x(t)$. In this way, we deduce that the composition $(M \circ x)(t)$ provides us with the magnitude M as a function of the variable t. For this reason, it is sometimes written in abbreviated form

$$M(t) = (M \circ x)(t) = \text{number of employees when the average monthly temperature is } t.$$

Example 49. For our fuel supply company, if we know that the average temperature for a certain month is t, it is possible to use the functions $x(t)$ and $M(x)$ in the following way

Known average temperature of a month $= t$	$\Rightarrow$	**We know the number of clients by means of** $= x(t)$	$\Rightarrow$	**And then, employees** $= M(x(t))$

In this way, the composition $(M \circ x)(t)$ provides us with the number of employees in the company as a function of the average monthly temperature.

If the function $M(x)$ that gives the number of employees as a function of the number of clients is given by the formula

$$M(x) = \frac{100x}{x + 100}$$

and the function $x(t)$ of the number of clients as a function of the average monthly temperature (measured in Celsius degrees) is

$$x(t) = \frac{10000}{t+30}$$

Then, the function that gives the number of employees as a function of the temperature will be given by the composition

$$(M \circ x)(t) = M(x(t)) = M\left(\frac{10000}{t+30}\right) = \frac{100\frac{10000}{t+30}}{\frac{10000}{t+30}+100} = \frac{10000}{t+130}$$

In short, we denote this function as $M(t)$ since it expresses M as a function of t. Thus,

$M(x) = \dfrac{100x}{x+100}$	expresses M (employees) as a function of x (clients)
$M(t) = M(x(t)) = \dfrac{10000}{t+130}$	expresses M (employees) as a function of t (temperature)

Next we see other similar examples in which functions representing different quantities or magnitudes as functions of different variables come into play.

Examples 50.

1) The function

$$p : [0, 365] \to \mathbb{R}$$
$$p(d) = \frac{d^2}{100} + 10d + 10000$$

provides the number of inhabitants (population) in a city for each day, d.

Suppose that the number of clients of a certain company in that city depends at each moment on the number of inhabitants according to the function

$$C : [0, \infty) \to \mathbb{R}$$
$$C(p) = p \cdot (0.1 + \tfrac{1}{p+1}) \; .$$

The function $C(p)$ gives us the number of clients as a function of the population of the city. If we want to obtain the number of clients, C, as a function of the day we are in, d, we must calculate the composition

$$(C \circ p)(d) = C(p(d)) = C\left(\frac{d^2}{100} + 10d + 10000\right) = \left(\frac{d^2}{100} + 10d + 10000\right)\left(0.1 + \frac{1}{(\frac{d^2}{100} + 10d + 10000) + 1}\right)$$
$$= \frac{1001100000 + 2001100d + 3001.1d^2 + 2d^3 + 0.001d^4}{1000100 + 1000d + d^2}.$$

Therefore, C as a function of d is

$$C(d) = (C \circ p)(d) = \frac{1001100000 + 2001100d + 3001.1d^2 + 2d^3 + 0.001d^4}{1000100 + 1000d + d^2}.$$

2) The average price (in euros) of a hundre square meters apartment, p, depends on the number of square meters of land available for construction, m, according to the function

$$p : (0, \infty) \to \mathbb{R}$$
$$p(m) = 60000 + \frac{10000000}{m} \; ,$$

On the other hand, the number of buyers who demand this type of apartments, D, depends on its price according to the function

$$D : [0, \infty) \to \mathbb{R}$$
$$D(p) = \frac{60000000}{p+1}\ .$$

To study how the demand (number of buyers) varies as a function of the available square meters, we must calculate the composition $D \circ p$ as follows:

$$(D \circ p)(m) = D(p(m)) = D\left(60000 + \frac{10000000}{m}\right) = \frac{60000000}{60000 + \frac{10000000}{m} + 1}.$$

In this way, the function that gives us the number of buyers who demand this type of apartments depends on the square meters offered for construction according to the function

$$D(m) = (D \circ p)(m) = \frac{60000000}{60000 + \frac{10000000}{m} + 1}.$$

In both items of **Examples 50** we have seen how the same magnitude can be expressed as a function of two different variables. For example, in part **2)** the demand of buyers, D, appears first as a function of the price of the dwellings, p, by means of the function $D(p)$ and then, after calculating the composition of $D(p)$ and $p(m)$, we obtain D as a function of the available meters, m, by means of the function $(D \circ p)(m)$. We have also seen that in abbreviated form we denote $(D \circ p)(m)$ as $D(m)$ since this composition really expresses the demand D as a function of m. However, it must be taken into account that this notation for the composition (that is, writing $(D \circ p)(m)$ as $D(m)$), although useful and more expressive in many occasions, is ambiguous since it might seem that we have two different formulas, $D(p)$ and $D(m)$, for the same function. If we were asked to calculate $D(2)$, initially we would not know whether we should apply the formula for $D(p)$, in which case

$$D(2) = \frac{60000000}{2+1} = 20000000,$$

or the formula for $D(m)$ which would give us

$$D(2) = \frac{60000000}{60000 + 5000000 + 1} = 11.8577$$

The correct application of this notation depends on the fact that we know precisely what units the quantity 2 corresponds to. If it is 2 euros we should apply $D(p)$ and we would obtain the first of the two results $D(2) = 20000000$. On the contrary, if we are talking about 2m^2, we would use $D(m)$ and then the correct option is the second one $D(2) = 11.8577$.

The Inverse Function

Given a function $f : I \subseteq \mathbb{R} \to f(I) \subseteq \mathbb{R}$, for every point $x \in D$, f transforms x into $y = f(x)$,

$$\begin{array}{rccc} f: & I & \to & f(I) \\ & x & \mapsto & y = f(x). \end{array}$$

We try now to find the function performing the inverse transformation, that is to say

$$\begin{array}{ccc} f(I) & \to & I \\ y = f(x) & \mapsto & x \end{array};$$

such a function would be the inverse function of f. To do so in a suitable way we need the function f to satisfy that for any $y \in f(I)$, we have only one possible $x \in I$ such that $f(x) = y$. We call injective to any function satisfying this property.

This idea lead us to the following definition.

Definición 1. Given an injective function $f : I \subseteq \mathbb{R} \to f(I) \subseteq \mathbb{R}$, we call inverse function of f, denoted f^{-1}, to the function

$$f^{-1}: \begin{array}{l} f(I) \to I \\ y \mapsto f^{-1}(y) = x \text{ such that } f(x) = y \end{array}.$$

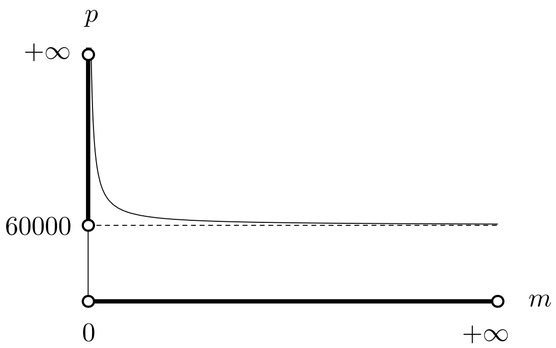
What interests us here is to give an interpretation of the concept of the inverse of a function.

To present these ideas we will study the function

$$\begin{array}{c} p : (0, \infty) \to \mathbb{R} \\ p(m) = 60000 + \dfrac{10000000}{m} \end{array}$$

that appears in part **2)** of **Examples 50** and that gives the average price of a 100m^2 dwelling, p, as a function of the number of meters available for construction, m.

If we observe the graph of the function p,



we can see at a glance that it is an injective function whose domain is the interval $(0, \infty)$. It is also observed that in this interval the values taken by $p(m)$ range from 60000 to $+\infty$, or equivalently

$$p\left((0, +\infty)\right) = (60000, +\infty).$$

This last point indicates that the function p can be written in the form

$$\begin{array}{c} p : (0, +\infty) \to (60000, +\infty) \\ p(m) = 60000 + \dfrac{10000000}{m} \end{array}.$$

If $p : (0, +\infty) \to (60000, +\infty)$ is injective, we can calculate its inverse function given by

$$\begin{array}{c} p^{-1} : (60000, +\infty) \to (0, +\infty) \\ p^{-1}(\mathbf{p}) = m \text{ such that } p(m) = \mathbf{p} \end{array}$$

That is, if p takes m to $\mathbf{p}$ then p^{-1} takes $\mathbf{p}$ to m:

$$m \underset{p^{-1}}{\overset{p}{\rightleftarrows}} \mathbf{p}$$

In this way, knowing the number of meters m, the function p gives us the price that the apartments will have and, conversely, knowing the price $\mathbf{p}$, the inverse p^{-1} gives as a result the number of meters, m, necessary to reach that price. In other words:

- The function $p(m)$ expresses the price, p, as a function of the available meters, m.
- The function $p^{-1}(\mathbf{p})$ expresses the meters, m, as a function of the price, $\mathbf{p}$.

For this reason, an appropriate notation for the inverse of the function $p(m)$ is $m(p)$ since in reality the inverse expresses the meters, m, as a function of the price, p. To calculate the inverse $m(p)$ we have that

$$p = p(m) = 60000 + \frac{10000000}{m} \Rightarrow \boxed{p = 60000 + \frac{10000000}{m}}$$

and, if we solve for m as a function of p ($m = m(p)$) in the boxed equality, we have $m = \frac{10000000}{p-60000}$. We thus obtain m as a function of p, that is, the inverse $m(p)$ which we can finally write in the form

$$\begin{aligned} m &: (60000, +\infty) \to (0, +\infty) \\ m(p) &= \frac{10000000}{p - 60000} \end{aligned} .$$

1.6.3 Fitting Using Trigonometric Functions

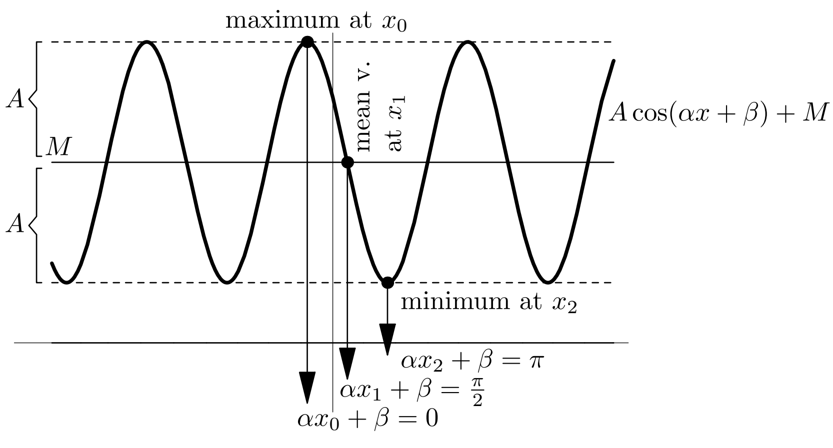
Extension of techniques with trigonometric functions. Page 17

The sine and cosine functions are periodic functions and therefore allow the mathematical modeling of certain cyclic phenomena in which a magnitude oscillates periodically around a mean value. To achieve this, we can use a trigonometric function of the type[1]

$$A\cos(\alpha x + \beta) + M.$$

The constants A, M, α, β that appear in the previous expression must be calculated in each case depending on the data of the phenomenon under study. To do so, we must take into account that:

- The constant M is the mean value around which the periodic function oscillates.
- The constant A is called the amplitude and indicates the maximum possible oscillation of the function with respect to the mean value (given by the constant M).
- To calculate the constants α and β, we must know that:
  - the function will reach its maximum value for $\alpha x + \beta = 0$,
  - it will reach its mean value for $\alpha x + \beta = \frac{\pi}{2}$,
  - and its minimum value when $\alpha x + \beta = \pi$.



[1]Actually, we could equally have used the generic function $A\,\mathrm{sen}(\alpha x + \beta) + M$.

Example 51. The amount of garbage produced daily in a city follows an annual cyclic behavior, oscillating around a mean value of 36500 tons/day. We know that the highest production of waste occurs at the end of the third month of the year (i.e., March), reaching then 50000 tons/day. Due to the population decrease during the summer holidays, the lowest production of garbage occurs at the end of the ninth month (i.e., September) and it is 23000 tons/day.

We can model the daily amount of garbage produced using a trigonometric function. We will use for this purpose a function of the type $A\cos(\alpha t+\beta)+M$ (we have called t the variable of the function since it designates the elapsed time). We only need to calculate the values of the constants A, M, α and β.

From the above statement it is easy to deduce the following information for the function we wish to obtain:

Period	12 Interval Interval $[0, 12]$	
Mean value	36500	
Minimum value	23000	reached for t=9
Maximum value	50000	reached for t=3
Maximum amplitude =Max valueue-Mean value =Mean value-Min valueue	13500	

From the table above, directly we extract the mean value, M, and the amplitude, A, so that the value of these constants will be

$$M = 36500 \qquad \text{and} \qquad A = 13500.$$

Let's calculate α and β:

- When the maximum value is reached, we have $\alpha t+\beta = 0$. But from the table it follows that this maximum value is obtained when $t = 3$. Therefore
$$\alpha \cdot 3 + \beta = 0.$$
- On the other hand, for the minimum value $\alpha t+\beta = \pi$ and we know from the table that this minimum value is reached for $t = 9$, so that
$$\alpha \cdot 9 + \beta = \pi.$$

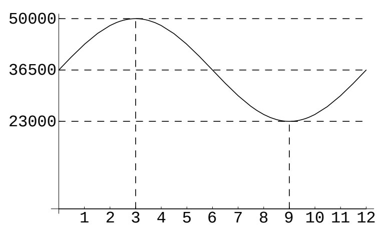
We thus obtain a system with two equations and two variables from which it is easy to solve for α and β:

$$\begin{cases} 3\alpha+\beta = 0 \\ 9\alpha+\beta = \pi \end{cases} \Rightarrow \alpha = \frac{\pi}{6}, \beta = -\frac{\pi}{2}.$$

With all this at hand, the trigonometric function that we will use to fit the situation of the problem is

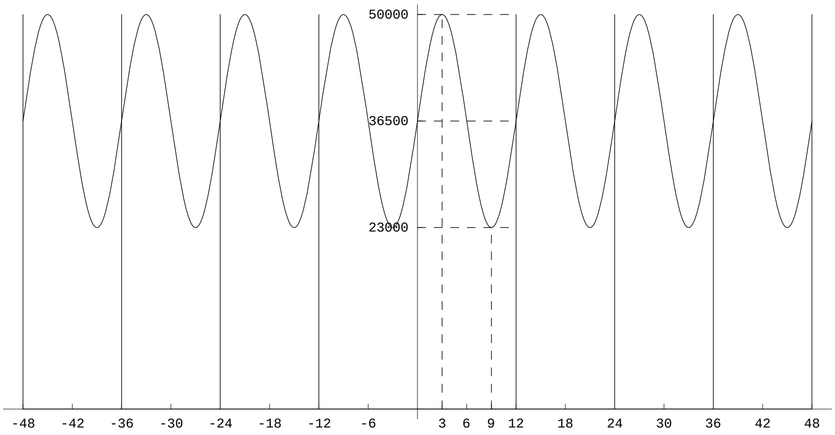
$$13500\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 36500.$$

which satisfies all our initial requirements since its graph fits the data of the problem:



$13500\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 36500$

Since the behavior in garbage production is the same every year, the graph of the function will repeat in each twelve-month segment:



1.6.4 Solving Trigonometric Equations Using Inverse Trigonometric Functions

Extension of techniques with inverse trigonometric functions. Page 18

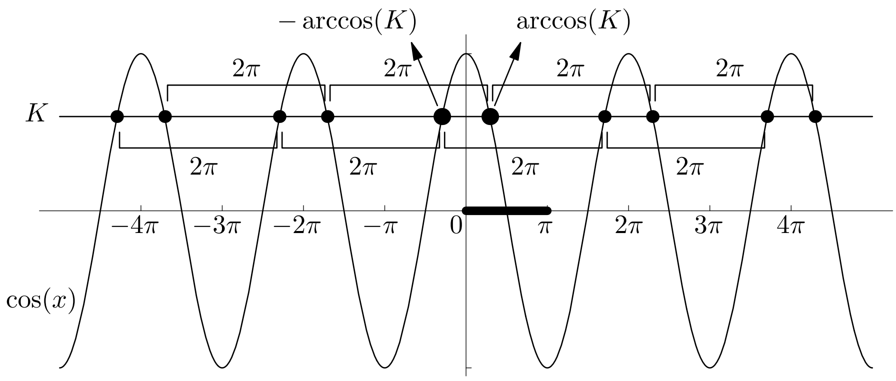
Suppose we want to solve the equation

$$\cos(x) = K$$

for certain value $K \in [-1, 1]$. We can solve for the value of x using the arccos function so that

$$x = \arccos(K).$$

However, this way we only obtain one solution when we know that this equation has infinitely many. How can we obtain all the other solutions?. In the graphical representation, we immediately realize that the solutions of the equation are determined by the intersections of the graph of $\cos(x)$ with the horizontal line at height K:



The only intersection we find within the interval $[0, \pi]$ (such an interval is marked with greater thickness in the graph) is the solution we obtain by calculating $\arccos(K)$ and we have represented it with a thicker point. Now, we observe in the graph that the value $-\arccos(K)$ (again represented with greater thickness) also represents a solution of the equation. Starting from these two points ($\arccos(K)$ and $-\arccos(K)$), if we advance or retreat in steps of length 2π we find new intersections that correspond to new solutions of the equation (these steps of length 2π starting from $\arccos(K)$ are represented in the graph by brackets above the line at height K and below it for those obtained from $-\arccos(K)$). In this way we obtain the solutions

$$\underbrace{\ldots, \arccos(K)-2(2\pi), \arccos(K)-2\pi, \arccos(K), \arccos(K)+2\pi, \arccos(K)+2(2\pi), \ldots}_{\text{starting from } \arccos(K)}.$$

$$\underbrace{\ldots, -\arccos(K)-2(2\pi), -\arccos(K)-2\pi, -\arccos(K), -\arccos(K)+2\pi, -\arccos(K)+2(2\pi), \ldots}_{\text{starting from } -\arccos(K)}.$$

That is to say, we get all the solutions that can be obtained from the following formulas by giving k different integer values ($k \in \mathbb{Z}$),

$$\arccos(K)+k(2\pi) \qquad \text{and} \qquad -\arccos(K)+k(2\pi).$$

We can summarize both expressions in the following one,

$$2k\pi \pm \arccos(K), \qquad k \in \mathbb{Z},$$

which collects all the possible solutions of the equation $\cos(x) = K$ that we wanted to solve initially.

Ejemplo 2. Suppose that in the situation described in **Example 51** we intend to determine in which months we have a garbage production of 40000 tons/day.

Earlier we took the function $13500\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 36500$ to fit the data of the problem using a trigonometric function. Now, to solve our problem, what we want to know is for which values of t such a function equals 40000. That is, we want to solve the equation

$$13500\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 36500 = 40000.$$

Solving for cos we obtain

$$\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) = \frac{3500}{13500} = \frac{7}{27}.$$

If we use what we have seen before to solve this equation, we will arrive at

$$\frac{\pi}{6}t - \frac{\pi}{2} = 2k\pi \pm \arccos\left(\frac{7}{27}\right)$$

and hence

$$t = \frac{2k\pi \pm \arccos\left(\frac{7}{27}\right) + \frac{\pi}{2}}{\frac{\pi}{6}} \Rightarrow t = 12k \pm \frac{6}{\pi}\arccos\left(\frac{7}{27}\right) + 3.$$

If we give different values to k we obtain

$$\begin{array}{lll}
\text{for } k=-1, & t = -12 \pm \frac{6}{\pi}\arccos\left(\frac{7}{27}\right) + 3 = \left\{\begin{array}{l} t = -6.50 \\ t = -11.49 \end{array}\right. , \\
\text{for } k=0, & t = \pm\frac{6}{\pi}\arccos\left(\frac{7}{27}\right) + 3 = \left\{\begin{array}{l} t = \underline{5.49} \\ t = \underline{0.50} \end{array}\right. , \\
\text{for } k=1, & t = 12 \pm \frac{6}{\pi}\arccos\left(\frac{7}{27}\right) + 3 = \left\{\begin{array}{l} t = 17.49 \\ t = 12.50 \end{array}\right. , \\
\text{for } k=2, & t = 24 \pm \frac{6}{\pi}\arccos\left(\frac{7}{27}\right) + 3 = \left\{\begin{array}{l} t = 29.49 \\ t = 24.50 \end{array}\right. .
\end{array}$$

Since we are only studying the 12 months of the year, only the values of t within the interval $[0, 12]$ will make sense (which appear underlined). It is easy to check that for other values of k we will no longer obtain new values of t between 0 and 12. Therefore, we finally deduce that the garbage production will reach 40000 tons/day precisely for $t = 0.50$ and for $t = 5.49$.

We can repeat the reasonings we have just done for arccos to conclude that, given $K \in [-1, 1]$, all the solutions of the equation

$$\text{sen}(x) = K$$

are given by the formulas

$$x = \text{arcsen}(K) + 2k\pi \qquad \text{and} \qquad x = \pi - \text{arcsen}(K) + 2k\pi, \qquad k \in \mathbb{Z}.$$

Similarly, if we intend to solve the equation

$$\tan(x) = K$$

for a certain value $K \in \mathbb{R}$, we can perform reasoning on the corresponding graphs that allows us to affirm that all the solutions of this equation are given by

$$x = \arctan(K) + k\pi, \qquad k \in \mathbb{Z}.$$

1.6.5 Analytical Definition of Limit

Extension of concepts about limits. Page 26

To understand this section, also see the additional material corresponding to neighborhoods on page 43.

In **Examples 28** it can be observed that lateral limits do not always make sense. There are points where only the limit from the left makes sense, in others only the limit from the right will make sense and finally there will also be points where both will make sense. In order to abbreviate we introduce the following notation:

Definición 3. Given the set $D \subseteq \mathbb{R}$:

i) We define the set $Ac^-(D)$ as that formed by the points $x \in \mathbb{R}$ such that for any $a < x$ we have $(a, x) \cap D \neq \emptyset$.

ii) We define the set $Ac^+(D)$ as that formed by the points $x \in \mathbb{R}$ such that for any $x < b$ we have $(x, b) \cap D \neq \emptyset$.

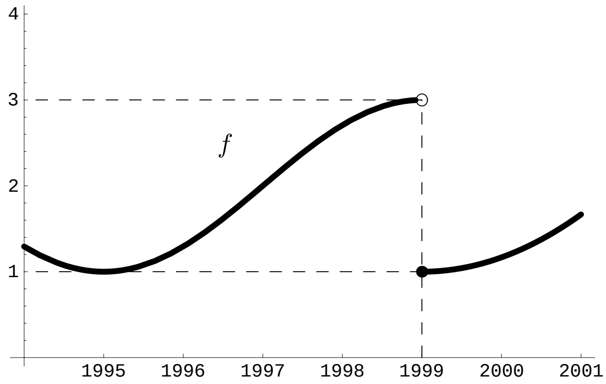
iii) $Ac(D) = Ac^-(D) \cup Ac^+(D)$.

Given the function $f : D \to \mathbb{R}$,

- the points of $Ac^-(D)$ are those where the limit from the left for f makes sense.
- the points of $Ac^+(D)$ are those for which the limit from the right makes sense.
- the points of $Ac^-(D) \cap Ac^+(D)$ are those for which both the limit from the left and from the right make sense.

Next we will give the rigorous definition of the limit of a function. Keep in mind that the definitions we have given on page 26 are of an intuitive type and have the purpose of introducing the definition we give now. We first present an example that justifies the definition of limit that we give later on page 55.

Ejemplo 4. The evolution of the income (in millions of euros) of a certain group of stock market investors is described from year 1994 on by the function f with graph:

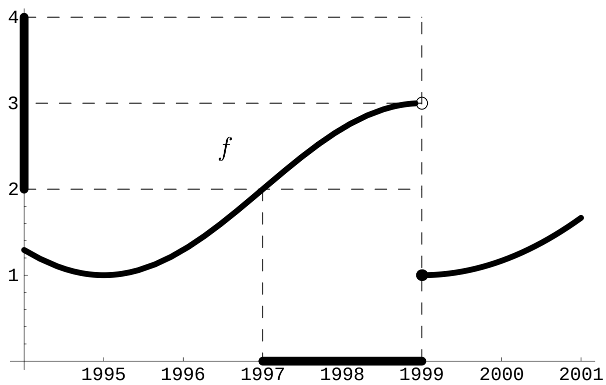


It is observed that at the beginning of 1999 the income drops sharply and from that moment on begins a slow recovery. Observing the graph it is clear that at the point $x_0 = 1999$ the values of the function to the left and right are different. That is, at said point the limits from the left and from the right have different values. Specifically, from the graph we deduce:

- Value of the function at the point: $f(1999) = 1$.
- Value of the function to the left of the point: $\lim_{x\to 1999^-} f(x) = 3$.
- Value of the function to the right of the point: $\lim_{x\to 1999^+} f(x) = 1$.

If we analyze the situation prior to the drop in income (i.e., the section of the function immediately before the point 1999) we observe that the profits were beginning to approach the amount of 3 million euros (the limit from the left at 1999 is 3) although it was never reached.

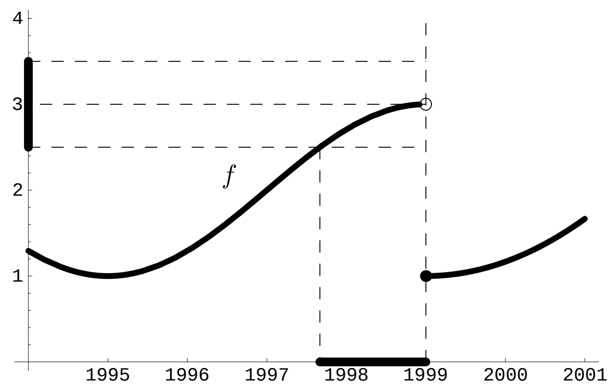
Let's take a neighborhood of the point 3 (remember that generally we work with neighborhoods instead of exact data) for example $(2, 4)$. Although the figure of 3 million was never reached, we can ask whether during some period immediately prior to the income collapse the profits moved within the selected neighborhood $(2, 4)$. Graphically we have:



It is observed that in the period comprised in the interval of years $(1997, 1999)$, the income moves within the neighborhood $(2, 4)$ of the point 3. In mathematical notation we will write this same thing as,

$$f((1997, 1999)) \subseteq (2, 4).$$

We can get closer to the limit quantity 3 by taking a smaller neighborhood. If we now choose the neighborhood $(2.5, 3.5)$ of 3 and try to find a period within which the income moves in this neighborhood, we have the following graphical situation:



Now, the graph indicates that within the period $(1997.67, 1999)$ the profits will be in the neighborhood $(2.5, 3.5)$ or in another way,

$$f((1997.67, 1999)) \subseteq (2.5, 3.5).$$

Successively we could narrow the neighborhood surrounding the limit quantity 3 and in each case look for a period of years in which the income is within these increasingly smaller neighborhoods. However the graph of the function and the previous examples seem to indicate that for any neighborhood I of 3, we find a period of years of the type $(a, 1999)$ within which the profits fall within said neighborhood. Written in mathematical format we have,

$$\forall I \text{ neighborhood of } 3, \exists\ a < 1999 \text{ such that } f((a, 1999)) \subseteq I.$$

Actually, this last mathematical condition constitutes a rigorous definition of what we have intuitively referred to as 'the value of f to the left of 1999 is 3' or as '$\lim\limits_{x \to 1999^-} f(x) = 3$'. In the following definition this same idea is extended to the general case.

It is evident that the study carried out to the left of the point 1999 can also be carried out to the right where we know that the limit is $\lim\limits_{x \to 1999^+} f(x) = 1$. In this case we will take neighborhoods that adjust to the limit value 1 and periods of time that are immediately after 1999, i.e., of the type $(1999, a)$.

Definición 5. Let the function $f : D \to \mathbb{R}$.

i) Given $x_0 \in \mathrm{Ac}^-(D)$, we say that the limit from the left of f at the point x_0 is $L \in \mathbb{R} \cup \{-\infty, +\infty\}$ and we denote it

$$\lim_{x \to x_0^-} f(x) = L,$$

if it holds that

$$\forall I \subseteq \mathbb{R} \text{ neighborhood of } L, \exists\ a < x_0 \text{ such that } f\left((a, x_0) \cap D\right) \subseteq I.$$

ii) Given $x_0 \in \mathrm{Ac}^+(D)$, we say that the limit from the right of f at the point x_0 is $L \in \mathbb{R} \cup \{-\infty, +\infty\}$ and we denote it

$$\lim_{x \to x_0^+} f(x) = L,$$

if it holds that

$$\forall I \subseteq \mathbb{R} \text{ neighborhood of } L, \exists\ a > x_0 \text{ such that } f\left((x_0, a) \cap D\right) \subseteq I.$$

iii) Given x_0, we say that the limit of f at the point x_0 is $L \in \mathbb{R} \cup \{-\infty, +\infty\}$ and we denote it

$$\lim_{x \to x_0} f(x) = L,$$

when:

- $x_0 \in \mathrm{Ac}^-(D) \cap \mathrm{Ac}^+(D)$, the limits from the right and from the left of f at x_0 exist and it holds that
$$\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x).$$
- $x_0 \in \mathrm{Ac}^-(D) - \mathrm{Ac}^+(D)$, the limit from the left of f at x_0 exists and
$$\lim_{x \to x_0^-} f(x) = L.$$
- $x_0 \in \mathrm{Ac}^+(D) - \mathrm{Ac}^-(D)$, the limit from the right of f at x_0 exists and
$$\lim_{x \to x_0^+} f(x) = L.$$

1.6.6 Continuously Compounded Interest

Extension of concepts about exponential type functions. Page 11

We will see how the results we have seen for calculating limits allow us to determine the formula for continuously compounded interest that we previously introduced without justification.

If we have a capital P and we invest it in an account with interest r compounded in m periods, we know that the capital function after t years is

$$P(t) = \left(1 + \frac{r}{m}\right)^{mt} P.$$

If we want to see what happens when the number of periods becomes very large (when $m \to \infty$) we will have to calculate the limit

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} P.$$

Note that in this expression both P and t and r are fixed quantities that depend on the data of the problem. In contrast m varies tending to infinity. It is evident that

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} P = P \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt}.$$

and therefore we only have to study the limit $\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt}$. If we calculate the limit by direct substitution and use the algebraic properties we obtain

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} = \left(1 + \frac{r}{\infty}\right)^{\infty t} = (1 + 0)^{\infty} = 1^{\infty}.$$

We obtain an indetermination of the type 1^{∞} which can be solved by **Property 37** in the following way: Instead of calculating the previous limit we transform it as indicated by the property into the form

$$\lim_{m \to \infty} mt \left(1 + \frac{r}{m} - 1\right) = \lim_{m \to \infty} mt \frac{r}{m} = \lim_{m \to \infty} rt = rt$$

so that applying **Property 37** we have that

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} = e^{rt}$$

and consequently

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} P = Pe^{rt}$$

so that the accumulated capital in year t at an interest rate r compounded continuously is

$$P(t) = Pe^{rt}.$$