Further Mathematics - Degree in Engineering - 2025/2026
02-Multivariate integral-Hand exam for serial number: 15

Exercise 3

Compute the volume of the domain given by $0 \le z \le 2\left(x^2 + y^2\right)$ and $\frac{1}{4}(-4+x)^2 + (-2+y)^2 \le 1$

1) 88.873
2) 1286.88
3) 267.035
4) 84.7643
5) 1199.16

We can write $\frac{1}{4}(-4+x)^2 + (-2+y)^2 \le 1$ as

$$\left(\frac{x-4}{2}\right)^2 + (y-2)^2 \le 1$$

So we have an elliptic cylinder with semi axes $a=2$, $b=1$ an centered at $(x_0, y_0) = (4,2)$ Therefore, we consider the transformation

$$T(\rho,\theta,z) = (\underbrace{4+2\rho\cos\theta}_{x}, \underbrace{2+\rho\sin\theta}_{y}, \underbrace{z}_{z})$$

with $|T'_{(\rho,\theta,z)}| = 2\rho$

The limits for variable z are given by the inequalities $0 \le z \le 2(x^2+y^2)$ If we apply the transformation to the upper paraboloid we have

$$2(x^2+y^2) \longrightarrow 2\left((4+2\cos\theta)^2+(2+\rho\sin\theta)^2\right)$$

As we know that in the elliptic cylinder ρ moves from 0 to 1, and θ from 0 to 2π, the integral for the volume is

$$\int_0^{2\pi}\int_0^1\int_0^{2\left((4+2\cos\theta)^2+(2+\rho\sin\theta)^2\right)} 2\rho \, dz \, d\rho \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2\rho \left[z\right]_0^{2\left((4+2\cos\theta)^2+(2+\rho\sin\theta)^2\right)} d\rho \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 4\rho\left((4+2\rho\cos\theta)^2+(2+\rho\sin\theta)^2\right) d\rho \, d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 \left(20\rho + 4\rho^3\cos^2\theta + 16\rho^2\cos\theta + \rho^3\sin^2\theta\right.$$

$$\left. + 4\rho^2\sin\theta\right) d\rho \, d\theta$$

$$= 4\int_0^{2\pi}\left[10\rho^2 + \rho^4\cos^2\theta + \frac{16}{3}\rho^3\cos\theta + \frac{1}{4}\rho^4\sin^2\theta + \frac{4}{3}\rho^3\sin\theta\right]_0^1 d\theta$$

$$= 4\int_0^{2\pi}\left(10 + \underbrace{\cos^2\theta}_{\frac{1}{2}+\frac{\cos(2\theta)}{2}} + \frac{16}{3}\cos\theta + \frac{1}{4}\underbrace{\sin^2\theta}_{\frac{1}{2}-\frac{\cos(2\theta)}{2}} + \frac{4}{3}\sin\theta\right)d\theta$$

$$= 4\left[10\theta + \frac{1}{2}\theta + \frac{\sin(2\theta)}{4} + \frac{16}{3}\sin\theta + \frac{1}{8}\theta - \frac{\sin(2\theta)}{16} + \frac{4}{3}\cos\theta\right]_0^{2\pi}$$

$$= 4\left(10\,2\pi + \frac{1}{2}2\pi + \frac{1}{8}2\pi\right) =$$

$$= 80\pi + 4\pi + \pi = 85\pi = 267'035$$