Hypersurfaces with constant higher order mean curvature

Barbara Nelli - Università dell'Aquila

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- H_1 is the mean curvature, H_2 is the scalar curvature, H_n is the Gauss-Kronecker curvature of $\mathbf{x}(M^n)$.

The higher order mean curvatures $\mathbf{H}_{\mathbf{r}}s$:

the origin and the geometry they carry.

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- In fact, R. Reilly gives a similar characterization in space forms.

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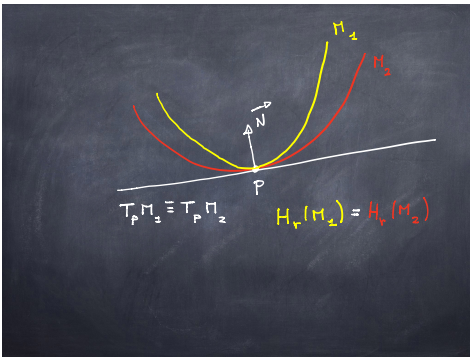
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- Such equation, for $H_r > 0$, is elliptic, provided the vectors of the principal curvatures belongs to Γ_r, the component of $H_r > 0$ in $\mathbb{R}^n$ containing $(1, \ldots, 1)$.
- $H_r = 0$ does not yield an elliptic equation in general (except when $r = 1$).

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Theorem [Hounie-Leite, JDG-1995, Fontenele-Silva, IJM 2001, AABS-2002]

Let M, M' be hypersurfaces in a Riemannian manifold $\mathcal{N}^{n+1}$ with the same H_r for some $1 \leq r \leq n$ and let $p \in M \cap M'$ such that $T_pM \equiv T_pM'$, $\mathbf{N}(p) = \mathbf{N}'(p)$. If M remains on one side of M' around p, then M and M' must coincide in a neighbourhood of p, provided:

- $H_r > 0$ and $\vec{\kappa}(p)$, $\vec{\kappa}'(p)$ belong to Γ_r.
- $H_r = 0$ and $\vec{\kappa}(p)$, $\vec{\kappa}'(p)$ belong to the same component of $H_r = 0$ and the rank of the Gauss map of either M or M' is $\geq r$.

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- The rank of the Gauss map coincide with the number of κ_i's different from zero or $H_{r+1} \neq 0$.
- The proof relies on the properties of the hyperbolic polynomials Garding [JMM 1959].

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- Caffarelli-Nirenberg-Spruck ['80's] studied existence and uniqueness of solutions for the Dirichlet problem for $H_r = cost > 0$ in $\mathbb{R}^{n+1}$ (continuity method: *a-priori* estimates for solutions of fully non linear PDE + existence of a subsolution).

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- N-Rosenberg, 1995, Alias-De Lira-Malacarne 2006 : H_r-hypersurfaces with boundary in Space Forms and Elbert-N-Santos, 2016 in $\mathbb{H}^n \times \mathbb{R}$.

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- **Stability.** The variational approach is not available for r-mean curvature hypersurfaces in a general Riemannian manifold, then we need an alternative notion of stability.

$$\mathbf{H_r = constant > 0} \text{ in } \mathbb{H}^{n+1}$$

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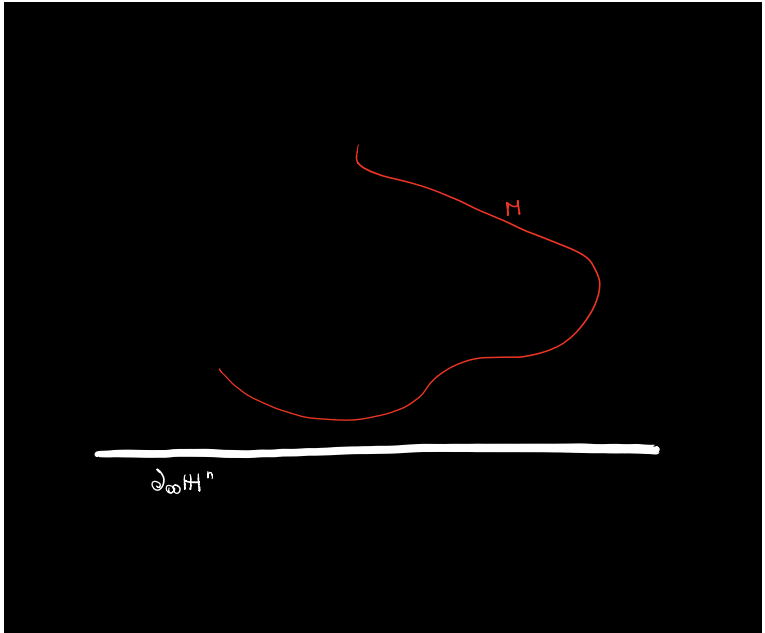
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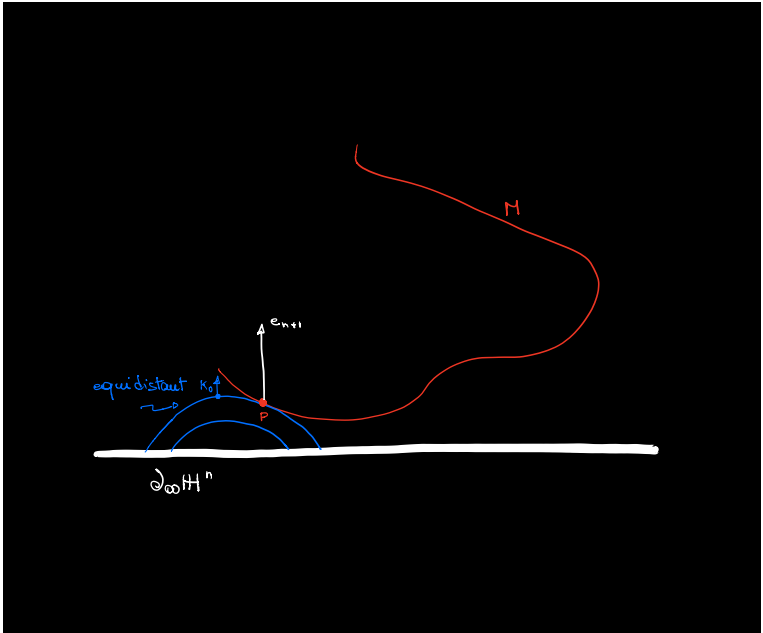
- This generalizes Do Carmo-Lawson [DMJ, 1983]).

Proof. Consider the case where $\partial_\infty M$ is a point. There exists a strictly convex point on M

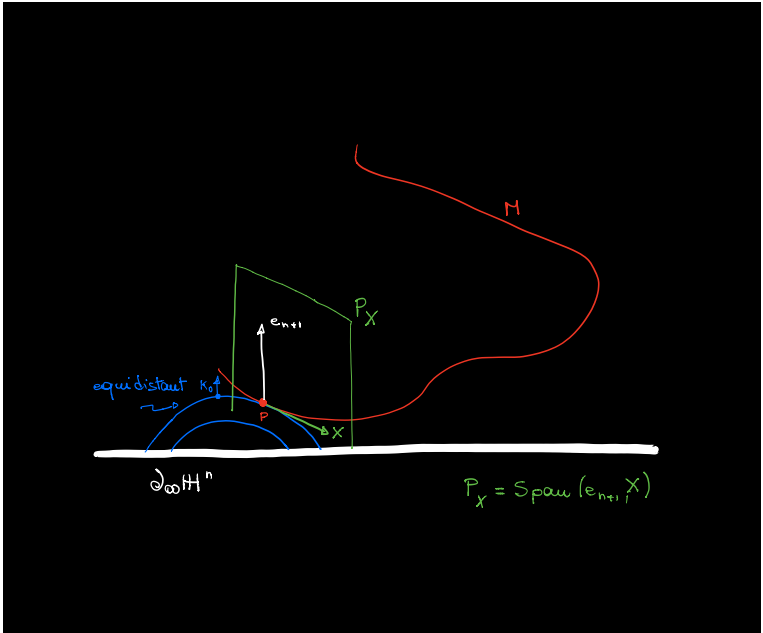
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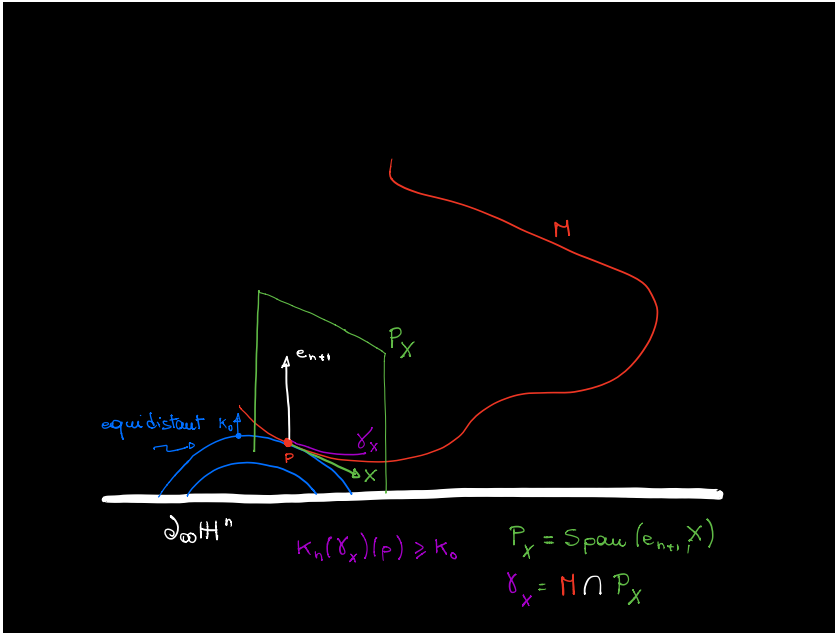
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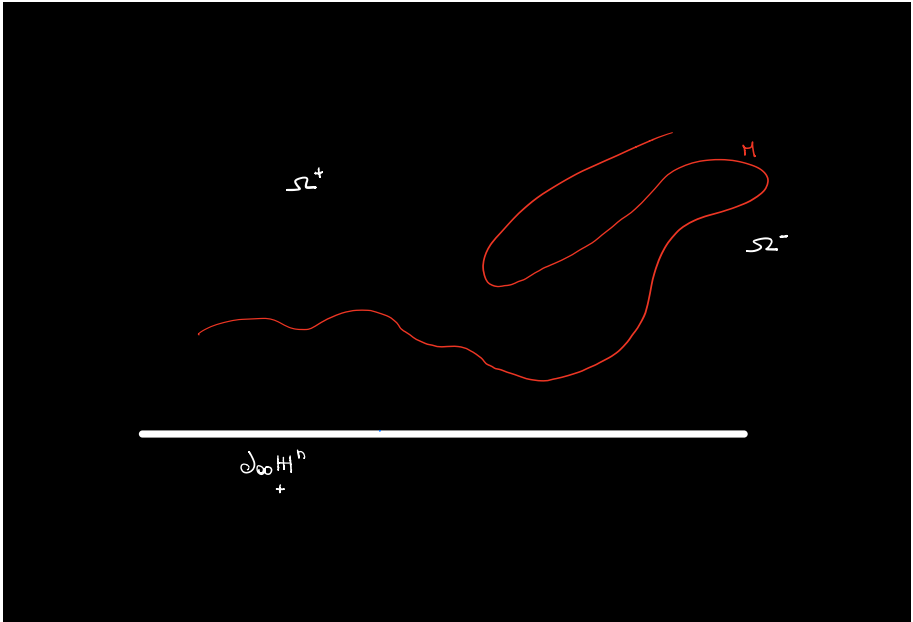


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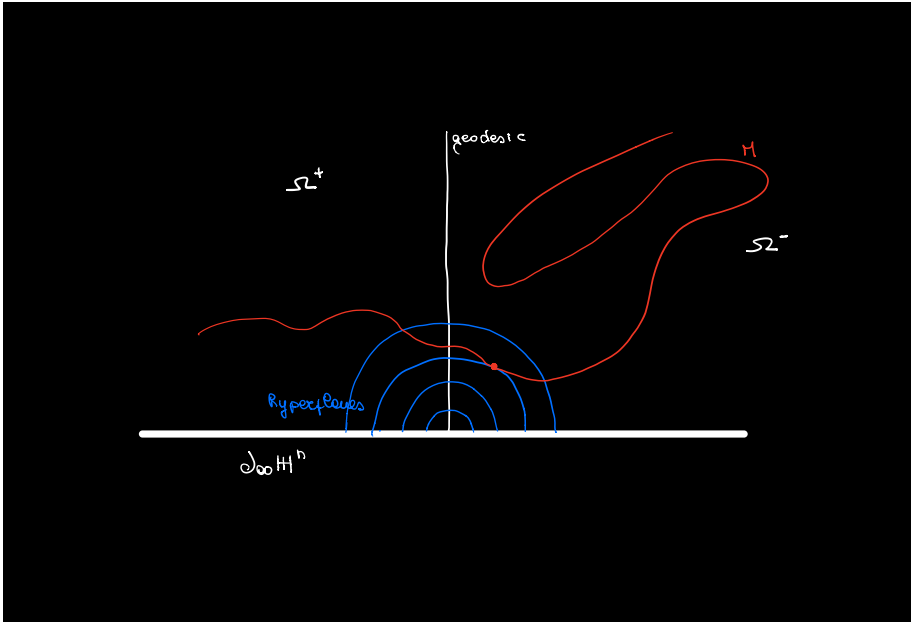


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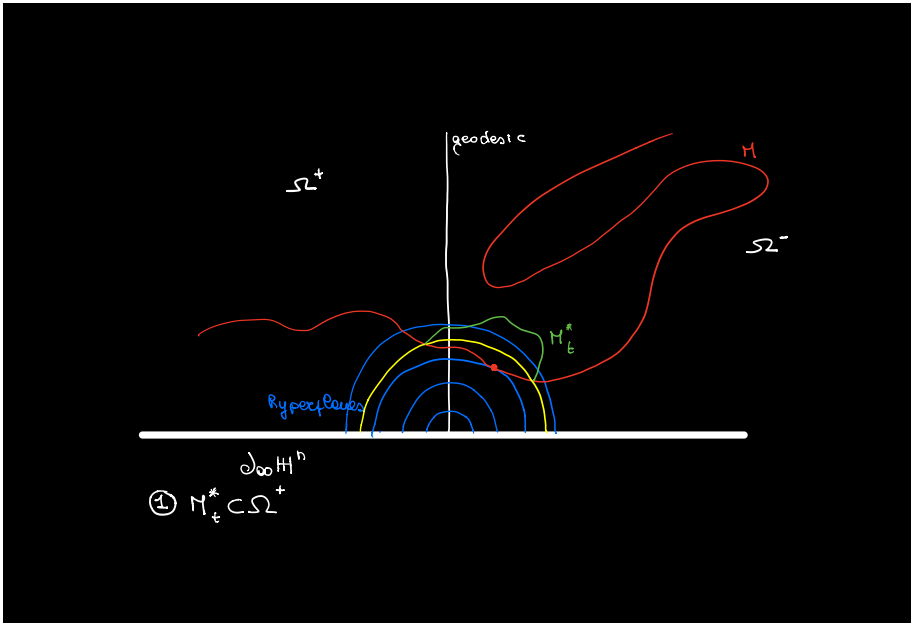
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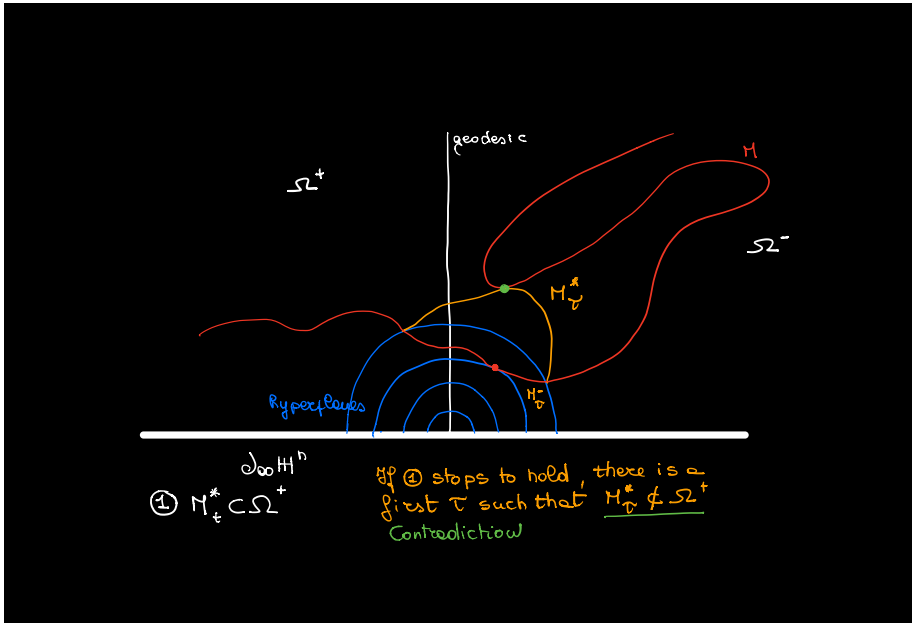
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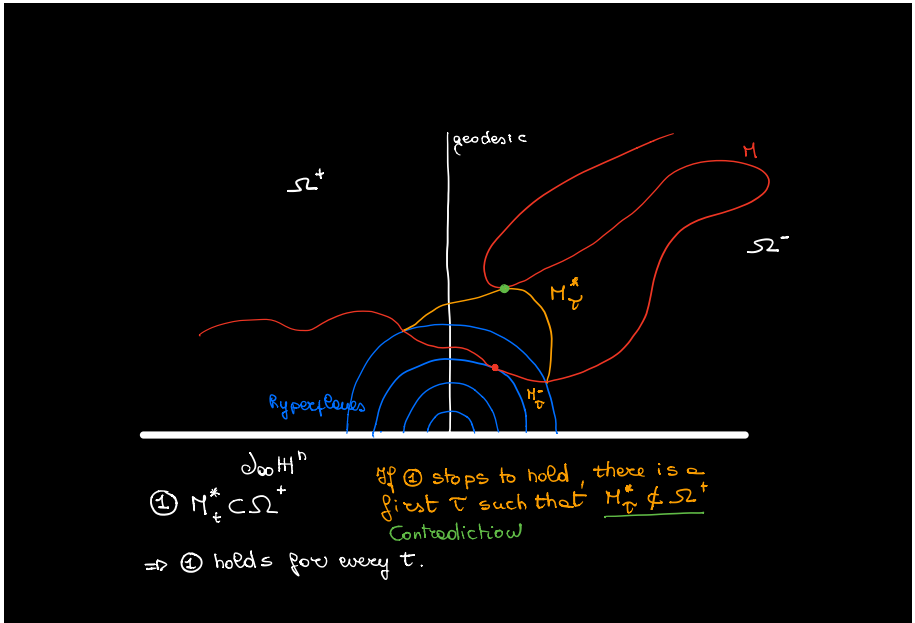
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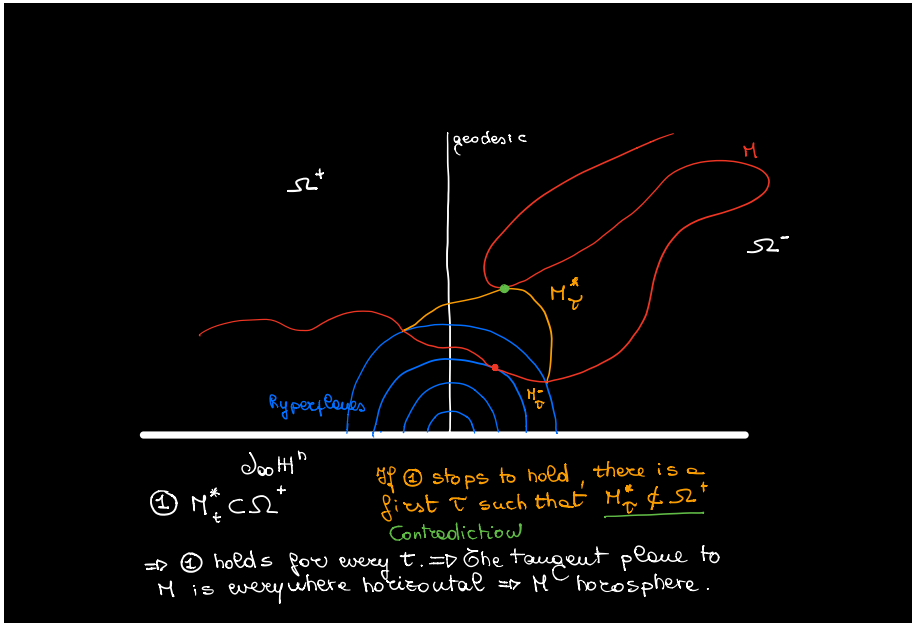
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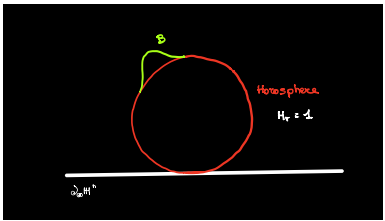
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A Rigidity result in $\mathbb{H}^{n+1}$

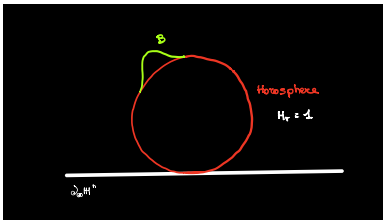
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Let Σ be a connected, properly embedded hypersurface in $\mathbb{H}^{n+1}$, which coincides with a horosphere outside a compact set **B** *with r-mean curvature $H_r = 1$ outside* **B**. *Then, if either $H_r \geq 1$ or $|H_r| \leq 1$ on Σ, then Σ is a horosphere.*



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- The same holds for equidistant spheres.
- The case of mean curvature is in Gromov [AIF, 2019] in the Euclidean space and Souam [JGA, 2021] in the hyperbolic space.

$$\mathbf{H_r} = \mathbf{0} \text{ in } \mathbb{H}^n \times \mathbb{R}$$

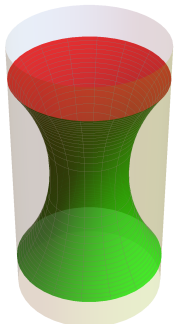
$\mathbf{H_r = 0}$ in $\mathbb{H}^n \times \mathbb{R}$

- We revisit two classical results for minimal surfaces with finite number of ends: Osermann, Schoen.

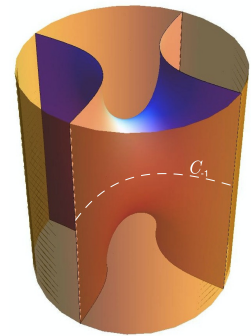
Two ends minimal examples in $\mathbb{H}^2 \times \mathbb{R}$

Two ends minimal examples in $\mathbb{H}^2 \times \mathbb{R}$

Vertical Catenoid



Horizontal Catenoid [Morabito, Rodriguez, 2012 JIMJ, Pyo, 2011 AGAG]



Courtesy of Martin-Mazzeo-Rodriguez

$H_r = 0$ IN $\mathbb{H}^n \times \mathbb{R}$: SCHOEN TYPE THEOREM.

THEOREM ELBERT-N-SANTOS [MANUSCRIPTA, 2016]

Let M be a complete, connected, embedded hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with $H_r = 0$, $2 \leq r < n$ with $H_{r+1} \neq 0$. Assume that M has two ends, each a vertical graph whose asymptotic boundary is a copy of $\partial_\infty \mathbb{H}^n$. Then, M is isometric, by an ambient isometry to a rotational invariant hypersurface.

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- The result is inspired by the well known theorem for minimal surfaces.

THEOREM SCHOEN [JDG, 1983]

A complete and connected minimal surface immersed in $\mathbb{R}^3$ with two embedded ends of finite total curvature is a catenoid.

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- The minimal case in $\mathbb{H}^n \times \mathbb{R}$ is studied in N-Sa Earp-Toubiana [ASNS, 2015].
- In Elbert-N-Santos [Manuscripta, 2016] r- minimal rotationally invariant hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ are classified.

$H_r = 0$ IN $\mathbb{H}^n \times \mathbb{R}$: OSSERMAN THEOREM.

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Let M be a complete oriented, immersed minimal surface in $\mathbb{R}^3$ **with finite total curvature**. *Then M is conformally equivalent to a compact Riemann surface with a finite number of points removed (the ends of M). Moreover, the Gauss map extends meromorphically to the punctures.*

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- Anderson [Preprint IHES, 1985] extended the result to higher dimension (diffeo, smooth Gauss map).
- We study complete hypersurfaces M in $\mathbb{H}^n \times \mathbb{R}$ provided they satisfy a finite total curvature assumption.
- Then, we specialise to M with $H_r = 0$ for some $r = 1, \ldots, n$.

Finite Strong Total Curvature

FINITE STRONG TOTAL CURVATURE

THEOREM (ELBERT-N [BULL. LONDON MATH. SOC. 2019])

Let $\mathbf{x} : M \longrightarrow \mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, *be an orientable complete hypersurface with finite strong total curvature. Then:*

1. *The immersion* $\mathbf{x}$ *is proper.*
2. M *is diffeomorphic to a compact manifold* $\bar{M}$ *minus a finite number of points* $q_1, \ldots, q_k$.

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DEFINITION.

Let A be the shape operator of M. We define $|| \, |A| \, ||_{W^{1,q}_{-1}(M)}$ to be the *strong total curvature* of M and we say that M has *finite strong total curvature* if $|A| \in W^{1,q}_{-1}(M)$, for some $q > n$, that is (ξ is the intrinsic distance from a fixed point of M)

$$|| \, |A| \, ||_{W^{1,q}_{-1}(M)} = \left(\int_M |A|^q \xi^{q-n} \, dM \right)^{1/q} + \left(\int_M |\nabla |A||^q \xi^{2q-n} \, dM \right)^{1/q} < \infty$$

FINITE STRONG TOTAL CURVATURE

- The norm

$$|| u ||_{W^{1,q}_{-1}(M)} = \left(\int_M u^q \xi^{q-n} \, dM \right)^{1/q} + \left(\int_M |\nabla u|^q \xi^{2q-n} \, dM \right)^{1/q}$$

  was used by Bartnik [CPAM, 1986], in a pioneer paper, to define a suitable decay at infinity of the metric of a manifold (asymptotically flat spaces) that guarantees that the ADM-mass is a geometric invariant.

- The notion of strong total curvature, was first introduced by Do Carmo-Elbert [CAG, 2019] for hypersurfaces in $\mathbb{R}^{n+1}$.

Finite strong total curvature and $H_r = 0$

THEOREM (ELBERT-N [BULL. LONDON MATH. SOC. 2019])

Let $X : M \longrightarrow \mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, be an orientable complete hypersurface with finite strong total curvature and $H_r = 0$. Let E be a punctured neighbourhood of one of the q_i' s and $N = (N_1, \ldots, N_{n+1})$ be a unit normal vector field on E. Let P be a hyperplane of $\mathbb{H}^n \times \mathbb{R}$ such that ∂E is contained in one of the half-spaces determined by P. Suppose that $\partial_\infty E \cap (\partial_\infty \mathbb{H}^n \times \mathbb{R}) \subset \partial_\infty P$. Then:

- *E is asymptotically close to P*
- *For any sequence of points $\{p_m\} \subset E$ converging to a point in $\partial_\infty E$, the sequence $\{N_{n+1}(p_m)\}$ converges uniformly to zero.*

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- The result generalises Hauswirth-Rosenberg [Mat. Cont. 2006], Hauswirth-N-Sa Earp-Toubiana [Adv. Math. 2015] while the proof takes inspiration in Sa Earp-Toubiana [JLMS 2015] ($H = 0$ in $\mathbb{H}^2 \times \mathbb{R}$)

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- When working with $n > 2$ one loses the technical support of the complex analysis and with $r > 1$ one weakens the technical support given by the theory of quasi-linear PDE. Then it seems reasonable to require a stronger assumption on the total curvature.

Stability.

- The immersion $\mathbf{x} : M \longrightarrow \mathbb{R}^{n+1}$ is *r-stable* if it is a local minimum for the functional $J_r = A_r + \binom{n}{r+1} H_{r+1} V$, $(J_r''(0) \geq 0)$.

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- When the ambient is a general Riemannian manifold $\mathcal{N}$, there is not known functional for which hypersurfaces with constant H_{r+1} are solutions of the associated variational problem.
- One can compute the linearized operator: $T_r := \binom{n}{r+1}^{-1} \frac{\partial}{\partial t} H_{r+1}$

$$T_r(f) = div(P_r \nabla f) - \langle tr(\nabla P_r), \nabla f \rangle + f(tr(A^2 P_r) + tr(P_r R_{\mathcal{N}})).$$

  where $f \in C^\infty(M)$, P_r is the classical Newton transformation defined inductively by $P_0 = I$, $P_r = \binom{n}{r} H_r - A P_{r-1}$ (A is the shape operator), $R_{\mathcal{N}}$ the curvature of $\mathcal{N}$.
- Notice that $tr(\nabla P_r) = 0$ in Space Forms.

Stability

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- This notion comes from the caracterization by [Fischer Colbrie-Schoen, CPAM 1980] and was used by [Bueno-Galvez-Mira, TAMS 2020], [Andersson-Eichmar-Metzger, JDG 2010].

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- We say that M is (weakly) *symmetric r-stable* if $-\int_M f \mathcal{T}_r^s f \geq 0$, $f \in C_0^\infty(M)$ ($\int_M f = 0$).

Three stability notions

M is r-stable
$T_r(u) \leq 0$ $u \in C^\infty(M)$, $u > 0$.
$(\lambda_p \geq 0)$

M symmetric r-stable
$-\int_M f T_r^s f \geq 0$, $f \in C_0^\infty(M)$
$(\lambda_1 \geq 0)$

M weakly symmetric r-stable
$-\int_M f T_r^s f \geq 0$, $f \in C_0^\infty(M)$
$\int_M f = 0$

- Which are the basic and suitable questions about r-stability and (weakly) symmetric r-stability?

In progress

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- **Question.** Under which condition on the constant H_r, complete r-stable hypersurfaces of a Riemannian manifold do not exist?

  In the case of constant mean curvature ($r = 1$) and low dimension, an answer is given by [Cheng, ArM 2006], [Elbert-N-Rosenberg, PAMS 2007]: they do not exist provided the mean curvature is large enough.

Nesting property

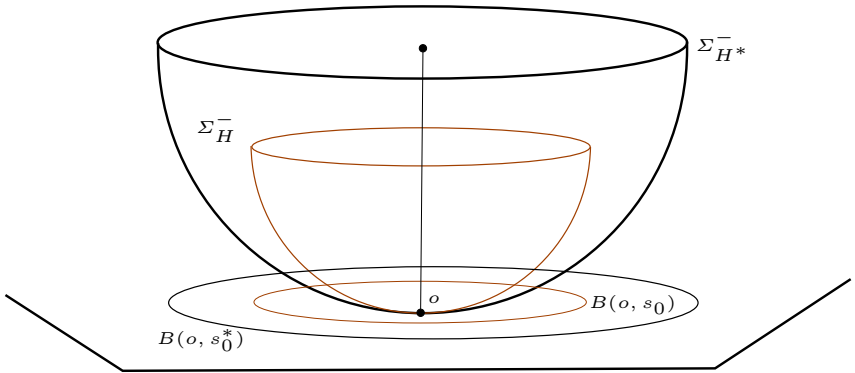
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Let $\mathcal{F}^r_{-1}$ (resp. $\mathcal{F}^r_1$) be the family of compact rotational hyperspheres Σ_{H_r} with constant H_r curvature in $\mathbb{H}^n \times \mathbb{R}$ (resp. $\mathbb{S}^n \times \mathbb{R}$), with the following properties:

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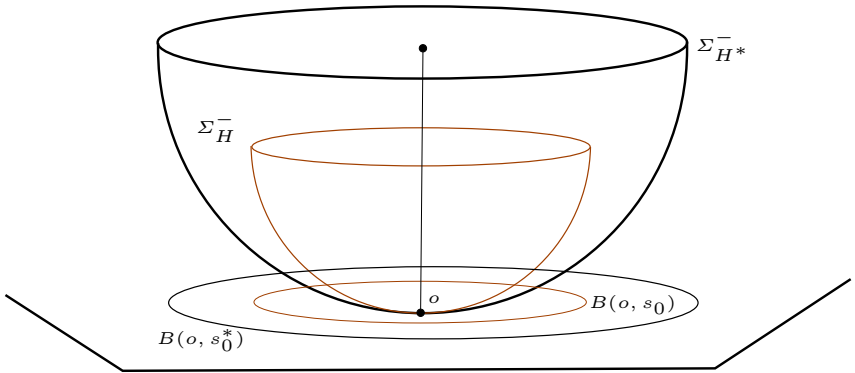
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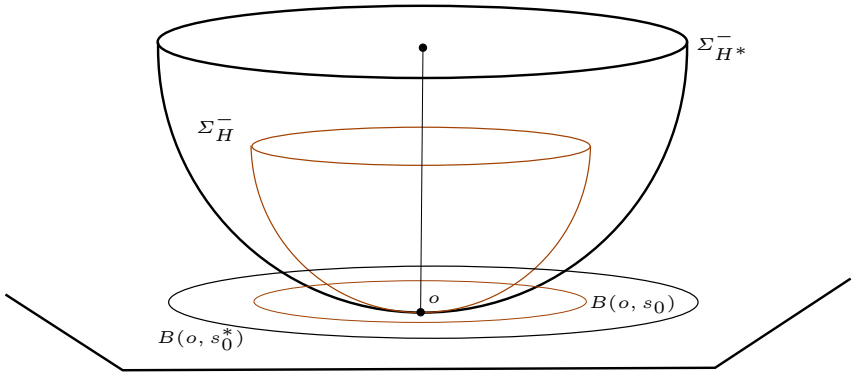
- $\{o\} \times \mathbb{R}$ is the axis of rotation of all spheres of $\mathcal{F}^r_{\pm 1}$.
- o is the point of minimal height of any Σ_{H_r} and the horizontal hyperplane $\mathbb{H}^n \times \{0\}$ (resp. $\mathbb{S}^n \times \{0\}$) is tangent to any such Σ_{H_r} at o.



Given $\Sigma_{H_r} \in \mathcal{F}^r_{\pm 1}$, denote by Ω_{H_r} the compact set bounded by the sphere Σ_{H_r}.

THEOREM ([ELBERT-FREIRE-N, IN PROGRESS])

Given $r \in \{1, \dots, n\}$, there exists a nonnegative constant $\alpha(\epsilon, r)$, $\epsilon = \pm 1$, such that, whenever $\Sigma_{H_r}, \Sigma_{H^*_r}$ satisfy $H_r > H^*_r > \alpha$, one has $\Omega_{H_r} \subset \Omega_{H^*_r}$. Moreover, if either $\epsilon = -1$ and $1 \leq r \leq n$, or $\epsilon = 1$ and $1 < r < n$, then the whole family is nested.



Gracias