Embedded free boundary CMC annuli in the unit ball

$\mathcal{H}$-Workshop 2023

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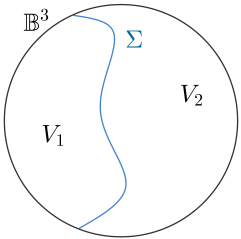
Joint work with Isabel Fernández and Pablo Mira

Introduction

A constant mean curvature (**CMC**) surface is **free boundary** in the unit ball $\mathbb{B}^3$ if it intersects orthogonally the boundary of $\mathbb{B}^3$.

Partitioning Problem

Critical points of the area functional among all surfaces that divide $\mathbb{B}^3$ into two pieces of prescribed volumes.



Introduction

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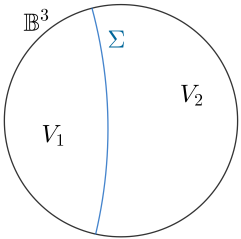
Partitioning Problem

Critical points of the area functional among all surfaces that divide $\mathbb{B}^3$ into two pieces of prescribed volumes.

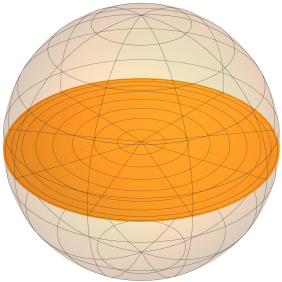
Solutions to this problem are **embedded**, free boundary CMC surfaces in $\mathbb{B}^3$.

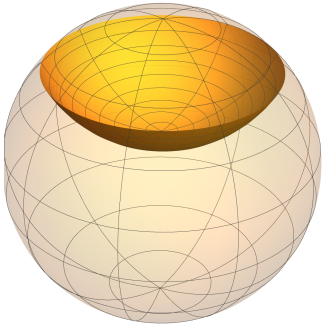


Topological Uniqueness

Theorem [Nitsche, 1985]

Any free boundary CMC disk in $\mathbb{B}^3$ is either an equatorial disk or a spherical cap.

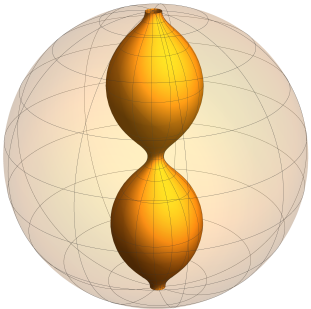


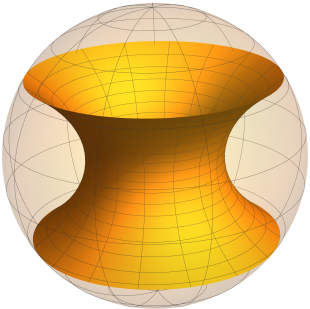


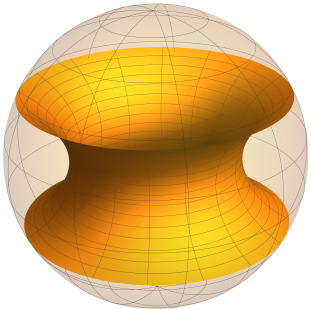
Annular case

Nitsche (1985)

Are Delaunay surfaces the only free boundary CMC annuli in $\mathbb{B}^3$?







Annular case

Theorem [Wente, 1995]

There are **immersed** free boundary CMC ($H \neq 0$) annuli in $\mathbb{B}^3$ which are **not rotational**.

Wente (1995)

Are Delaunay surfaces the only **embedded** free boundary CMC annuli in $\mathbb{B}^3$?

Theorem [Fernández-Hauswirth-Mira, 2022]

There are immersed free boundary minimal annuli in $\mathbb{B}^3$ which are **not rotational**.

Annular case

Theorem [C.-Fernández-Mira, 2022]

There are **embedded** free boundary CMC ($H \neq 0$) annuli in $\mathbb{B}^3$ which are **not rotational**.

More specifically,

For all $n \in \mathbb{N}$, $n \geq 2$, there is a real analytic family $\{\mathbb{A}_n(\mu) \,:\, \mu \in [0, \varepsilon)\}$ of embedded free boundary CMC annuli in $\mathbb{B}^3$ such that:

- $\mathbb{A}_n(0)$ is a compact embedded piece of a **nodoid**.
- If $\mu \neq 0$, $\mathbb{A}_n(\mu)$ is **not rotational**, and has a prismatic symmetry group $D_n \times \mathbb{Z}_2$.
- The annuli are foliated by spherical curvature lines.

Talk structure

1. CMC surfaces with spherical curvature lines
2. Symmetries and the annular condition
3. The spherical free boundary condition

CMC surfaces with spherical curvature lines (Wente)

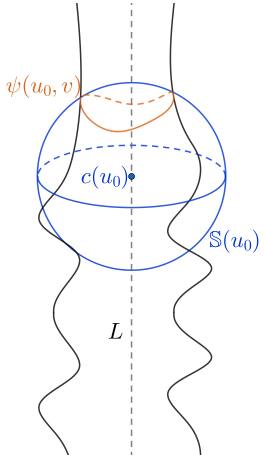
Let $\psi(u,v):\Omega\subset\mathbb{R}^2\to\mathbb{R}^3$ parametrize a CMC surface Σ with

1. $I = e^{2\omega}(du^2+dv^2)$ $\quad\Longrightarrow$ Conformal immersion.
2. $II = \cosh\omega\, du^2 + \sinh\omega\, dv^2 \Longrightarrow$ Param. by curvature lines
3. Each v-curvature line $v\to\psi(u_0,v)$ is **spherical**, i.e., it lies in a sphere $\mathbb{S}(u_0)$:

$$\|\psi(u_0,v)-c(u_0)\|^2 = R(u_0)^2.$$

Then,

- The intersection angle of Σ with the sphere is constant: $\theta=\theta(u)$. (Joachimsthal)
- The centers $u\to c(u)$ lie in a common (vertical) axis.



CMC surfaces with spherical curvature lines (Wente)

The previous geometric conditions translate into equations for $\omega(u, v) : \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{cases} \Delta\omega + \sinh\omega\cosh\omega = 0, \\ \omega_u = y(u)\cosh\omega + z(u)\sinh\omega. \end{cases}$$

This overdetermined system has solutions iff $(y(u), z(u))$ satisfy

$$\begin{cases} y'' = (\hat{a} - 1)y - 2y(y^2 - z^2), \\ z'' = \hat{a}z - 2z(y^2 - z^2). \end{cases}$$

Degrees of freedom: initial conditions $y(0), z(0), y'(0), z'(0)$ and the constant $\hat{a}$.

We fix $y(0) = z(0) = 0$.

We also define three parameters $(\alpha, \beta, \gamma) \in [1, \infty)^3$ in terms of the variables $(y'(0), z'(0), \hat{a})$.

Summary: given $(\alpha, \beta, \gamma) \in [1, \infty)^3$, we get an analytic solution $\omega(u, v) : \mathbb{R}^2 \to \mathbb{R}$.

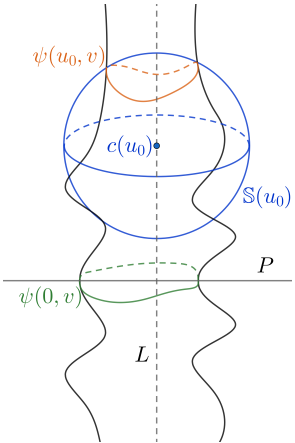
There is a unique CMC $H = \frac{1}{2}$ immersion $\psi(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$ with

- $I = e^{2\omega}(du^2 + dv^2)$.
- $II = \cosh\omega du^2 + \sinh\omega dv^2$.
- Each v-curvature line $v \to \psi(u_0, v)$ lies on a sphere $\mathbb{S}(u_0)$.

$$R(u)^2 = \frac{1 + (y - z)^2}{y^2}, \quad \tan(\theta(u)) = \frac{-1}{y - z}, \quad c'(u) = \frac{y'(0)}{y^2} e_3,$$

where e_3 is the vertical unit vector.

$y(0) = 0, z(0) = 0$: the curve $v \to \psi(0, v)$ is a planar geodesic. Σ is **symmetric** with respect to a horizontal plane.

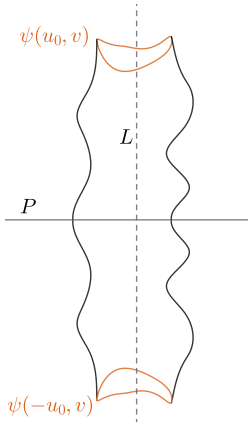


Embedded free boundary CMC annuli in $\mathbb{B}^3$

Given (α, β, γ), we have an immersion $\psi(u, v)$.

Now we restrict $\psi(u, v) : [-u_0, u_0] \times \mathbb{R} \to \mathbb{R}^3$. We want ψ to satisfy:

Annulus: $\psi(u, v)$ periodic in v-direction.



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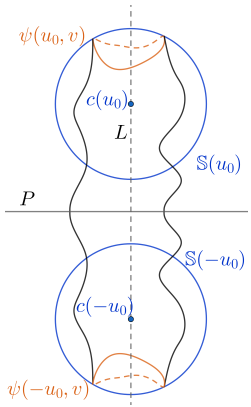
Spherical boundary: The two boundary spheres $\mathbb{S}(u_0)$, $\mathbb{S}(-u_0)$ coincide.



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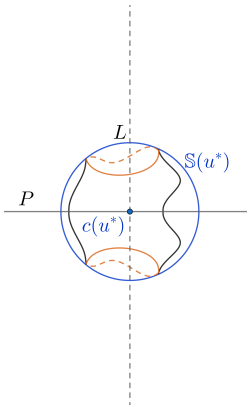
Spherical boundary: The two boundary spheres $\mathbb{S}(u_0)$, $\mathbb{S}(-u_0)$ coincide.

Free boundary: The intersection angle between $\mathbb{S}(u^*)$ and Σ is $\theta(u^*) = \frac{\pi}{2}$.

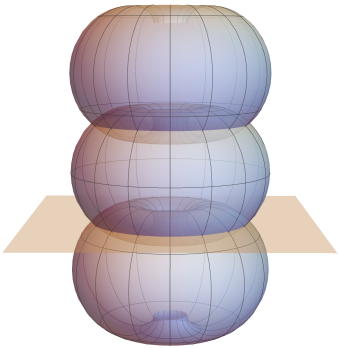
Embedded and inside the sphere $\mathbb{S}(u^*)$.



Degenerate case ($\alpha = 1$): nodoids

If $\alpha = 1$, $\psi(u, v)$ is **nodoid**.

- $u \to \psi(u, 0)$ is the profile curve of Σ.
- The curves $v \to \psi(u_0, v)$ are horizontal circles.
- The centers $c(u)$ lie in the rotation axis of Σ.



Talk structure

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Description of symmetries

We now let $\alpha > 1$. Σ satisfies:

- **Symmetry** with respect to the horizontal plane P. (known)
- **Symmetry** with respect to (at least) one vertical plane.
- Invariance under rotations of angle 2Θ, where $\Theta \in (0, \pi)$.

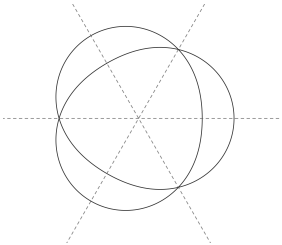
The map $\Theta = \Theta(\alpha, \beta, \gamma)$ is **analytic**.

Geometric interpretation of Θ. Annular condition

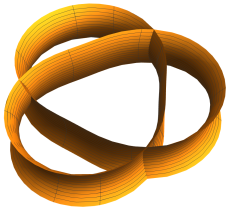
Σ is an **annulus** $\iff \psi(u,v)$ is v-periodic $\iff \Theta = \frac{m}{n}\pi$, where $m, n \in \mathbb{N}$, $0 < m < n$. Moreover,

- m is the rotation index of the curve $v \to \psi(0,v)$.
- n is the number of vertical planes of symmetry.

If $\Theta = \frac{2}{3}\pi$,



Planar geodesic $v \to \psi(0,v)$.



Immersion $\psi : [-u_0, u_0] \times \mathbb{R} \to \mathbb{R}^3$.

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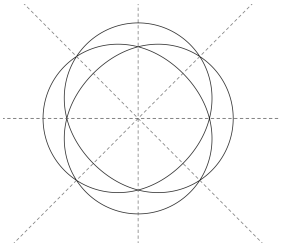
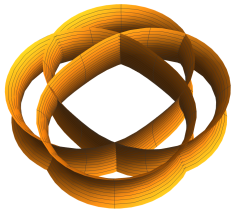
If $\Theta = \frac{3}{4}\pi$,



Planar geodesic $v \to \psi(0,v)$.



Immersion $\psi : [-u_0, u_0] \times \mathbb{R} \to \mathbb{R}^3$.

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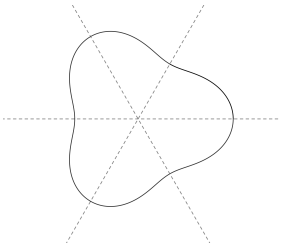
- m is the rotation index of the curve $v \to \psi(0,v)$.
- n is the number of vertical planes of symmetry.

Let $\Theta = \frac{1}{3}\pi$. If $m = 1$, we have **local embeddedness!**



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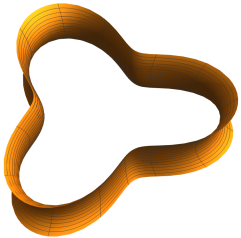

Immersion $\psi : [-u_0, u_0] \times \mathbb{R} \to \mathbb{R}^3$.

Annular condition

To obtain **embedded annuli**, we must restrict our study to those (α, β, γ) such that $\Theta(\alpha, \beta, \gamma) = \frac{\pi}{n}$, $n \in \mathbb{N}$, $n \geq 2$.

For every $n \geq 2$, the level set

$$\Theta(\alpha, \beta, \gamma) = \frac{\pi}{n}$$

can be seen as an **analytic graph** $\gamma_n = \gamma_n(\alpha, \beta)$.

From now on, we fix $n \geq 2$ and let $\gamma = \gamma_n(\alpha, \beta)$.

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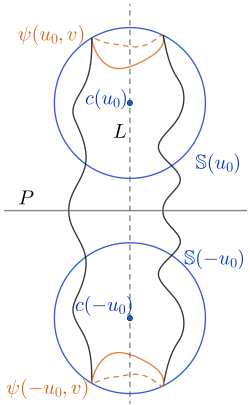
Spherical boundary

Let $(\alpha, \beta) \in [1, \infty)^2$ and $\gamma = \gamma_n(\alpha, \beta)$ so that $\psi(u, v)$ is an **annulus**. Take $u_0 > 0$. The restriction

$$\psi : [-u_0, u_0] \times \mathbb{R} \to \mathbb{R}^3$$

is a **compact annulus** whose boundary $\partial\Sigma$ intersects two spheres $\mathbb{S}(u_0)$, $\mathbb{S}(-u_0)$ with constant angle.



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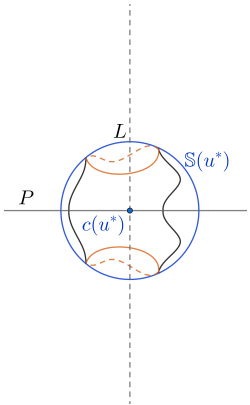
is a **compact annulus** whose boundary $\partial\Sigma$ intersects two spheres $\mathbb{S}(u_0)$, $\mathbb{S}(-u_0)$ with constant angle.

There exists $u^* > 0$ such that $c(u^*) \in L \cap P$. This implies that

$$\mathbb{S}(u^*) = \mathbb{S}(-u^*)$$

and the boundary $\partial\Sigma$ intersects $\mathbb{S}(u^*)$ with constant angle.

Moreover, the map $u^* = u^*(\alpha, \beta)$ is **analytic**.



Geometric meaning of u^*

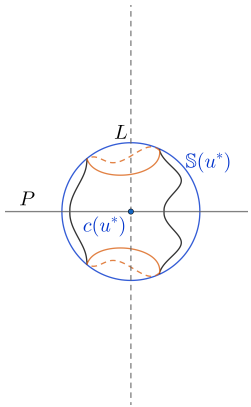
The restriction $\psi : [-u^*, u^*] \times \mathbb{R} \to \mathbb{R}^3$ satisfies:

- ✓ It is an annulus.
- ✓ It has spherical boundary.
- ✓ $\partial\Sigma$ intersects $\mathbb{S}(u^*)$ with constant angle $\theta(u^*)$, where

$$\tan(\theta(u^*)) = \frac{-1}{y(u^*) - z(u^*)}.$$

The free boundary condition is equivalent to $y(u^*) = z(u^*)$.

Does $y(u^*) = z(u^*)$ hold for some (α, β)?



The map $\tau(\alpha, \beta)$

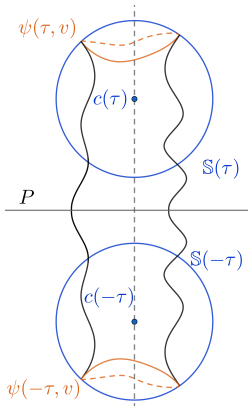
For every $(\alpha, \beta) \in \mathcal{W} \subsetneq [1, \infty)^2$ there exists $\tau > 0$ such that

$$y(\tau) = z(\tau).$$

Moreover, the map $\tau = \tau(\alpha, \beta)$ is **analytic**.

The annulus $\psi : [-\tau, \tau] \times \mathbb{R} \to \mathbb{R}^3$ satisfies:

- ✓ Its boundary components lie on spheres $\mathbb{S}(-\tau)$, $\mathbb{S}(\tau)$.
- ✓ $\partial\Sigma$ intersects the boundary spheres orthogonally.



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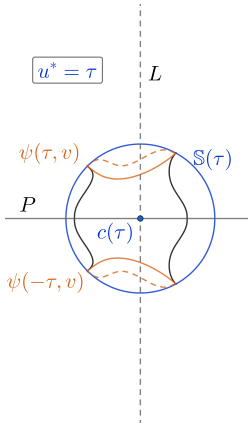
- ✓ Its boundary components lie on spheres $\mathbb{S}(-\tau)$, $\mathbb{S}(\tau)$.
- ✓ $\partial\Sigma$ intersects the boundary spheres orthogonally.

If $u^* = \tau$, then $\mathbb{S}(\tau) = \mathbb{S}(-\tau)$, and we obtain a free boundary annulus in $\mathbb{S}(\tau)$.



The free boundary condition: $u^* = \tau$

The map $u^*(\alpha, \beta)$ is defined for $(\alpha, \beta) \in [1, \infty)^2$.

However, $\tau(\alpha, \beta)$ is only defined on $\mathcal{W}$.

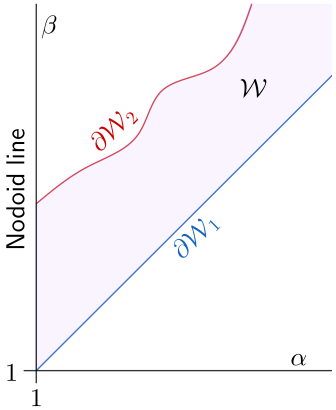
In a neighbourhood of $\alpha = 1$, $\mathcal{W}$ lies between two boundary components $\partial\mathcal{W}_1$, $\partial\mathcal{W}_2$.

If $(\alpha, \beta) \in \partial\mathcal{W}_1$,

$$u^*(\alpha, \beta) < \tau(\alpha, \beta).$$

If $(\alpha, \beta) \in \partial\mathcal{W}_2$,

$$u^*(\alpha, \beta) > \tau(\alpha, \beta).$$



The free boundary condition: $u^* = \tau$

There exists an analytic path $\Gamma(\mu) : [0, \varepsilon) \to \mathcal{W}$ such that $\Gamma(0) = (1, \beta^*)$ and

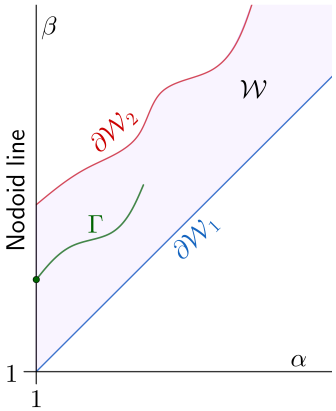
$$u^*(\Gamma(\mu)) = \tau(\Gamma(\mu))$$

for all $\mu \in [0, \varepsilon)$.

There is a 1-parameter analytic family of immersions $\Sigma(\mu) \equiv \psi(u, v) : [-u^*, u^*] \times \mathbb{R} \to \mathbb{R}^3$ satisfying:

- ✓ They are annuli ($\Theta = \frac{\pi}{n}$).
- ✓ Have spherical boundary since $c(u^*) = L \cap P$.
- ✓ Free boundary with respect to $\mathbb{S}(u^*)$, due to $\tau = u^*$.

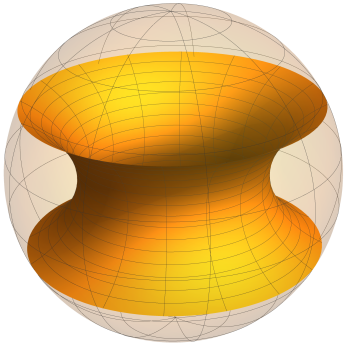
Are they embedded and in the interior of $\mathbb{S}(u^*)$?



Embeddedness

Since $\Gamma(0) = (1, \beta^*)$, the surface $\Sigma(0)$ is a **free boundary nodoid**.

The free boundary nodoid is **embedded** and lies inside the sphere $\mathbb{S}(u^*)$.



Embeddedness

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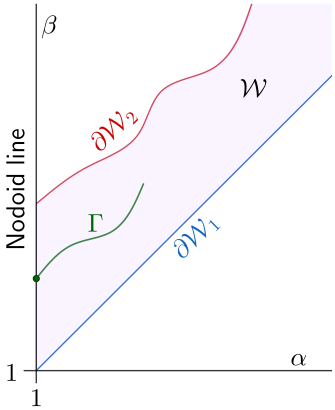
The free boundary nodoid is **embedded** and lies inside the sphere $\mathbb{S}(u^*)$.

For some $\varepsilon > 0$, the surfaces $\Sigma(\mu)$, $\mu \in [0, \varepsilon)$, are **embedded** and lie inside the sphere $\mathbb{S}(u^*)$ of radius $R(u^*)$.

If we denote by Ψ_μ the homothety that sends $\mathbb{S}(u^*)$ to the **unit sphere** $\mathbb{S}^2$,

The annuli $\mathbb{A}(\mu) := \Psi_\mu(\Sigma(\mu))$ are **embedded** and **free boundary** in the unit ball $\mathbb{B}^3$.



Theorem

Theorem [C.-Fernández-Mira, 2022]

There are **embedded** free boundary CMC ($H \neq 0$) annuli in $\mathbb{B}^3$ which are **not rotational**.

For all $n \in \mathbb{N}$, $n \geq 2$, there is a real analytic family $\{\mathbb{A}_n(\mu) \,:\, \mu \in [0, \varepsilon)\}$ of embedded free boundary CMC annuli in $\mathbb{B}^3$ such that:

- $\mathbb{A}_n(0)$ is a compact embedded piece of a **nodoid**.
- If $\mu \neq 0$, $\mathbb{A}_n(\mu)$ is **not rotational**, and has a prismatic symmetry group $D_n \times \mathbb{Z}_2$.
- The annuli are foliated by spherical curvature lines.

Thank you for your attention!

Cerezo, A., Fernández, I., Mira, P. *Annular solutions to the partitioning problem in a ball.*

