A Calabi-type correspondence for the prescribed mean curvature equation

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Abstract

We classify all Killing submersions from a Riemannian or Lorentzian orientable 3-manifold $\mathbb{E}$ to a simply-connected surface M in terms of a natural function $\tau \in \mathcal{C}^\infty(M)$, called bundle curvature. We present explicit models for such structures, enabling a generalization of the classical Calabi correspondence. Namely, it is a bijection between mean curvature H graphs in Riemannian Killing submersions with bundle curvature τ, and mean curvature τ spacelike graphs in Lorentzian Killing submersions with bundle curvature H (here H and τ are arbitrary smooth functions defined over a common base surface M). This leads to the existence of solutions for the prescribed mean curvature equation and to the non-existence of complete spacelike surfaces in a large class of spacetimes.

Killing submersions [2]

A *Killing submersion* is a Riemannian submersion $\pi : \mathbb{E} \to M$, where $\mathbb{E}$ is an orientable 3-manifold, M is an orientable Riemannian surface, and the fibers of π are the integral curves of a *unit* Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$.

- There exists a function $\tau \in \mathcal{C}^\infty(\mathbb{E})$, called *bundle curvature*, such that
  $$\overline{\nabla}_X \xi = \tau X \wedge \xi, \quad \text{for all } X \in \mathfrak{X}(\mathbb{E})$$
  It is constant along fibers $\rightsquigarrow \tau \in \mathcal{C}^\infty(M)$.
- All fibers share the same length.
- If M is not compact or the fibers have infinite length, then π admits a global section.

The submersion π is Riemannian if ξ is spacelike, otherwise it is called Lorentzian.

Classification over a disk

Theorem. Let M be a disk $\Omega \subset \mathbb{R}^2$ endowed with the metric $\lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)$, $\lambda > 0$. Given $\tau \in \mathcal{C}^\infty(M)$, any Killing submersion over M with bundle curvature τ, and whose fibers have infinite length, is isomorphic to

$$\pi_1 : \Omega \times \mathbb{R} \to \Omega, \quad \pi_1(x, y, z) = (x, y),$$

where $\Omega \times \mathbb{R}$ is endowed with the metric

$$\lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2) \pm (\mathrm{d}z \pm \mathbf{C}_{\lambda,\tau}(y\,\mathrm{d}x - x\,\mathrm{d}y))^2 \quad (\star)$$
$$\mathbf{C}_{\lambda,\tau} = 2\int_0^1 s\tau(xs, ys)\lambda(xs, ys)^2\,\mathrm{d}s.$$

The $\pm$ sign is chosen positive in the Riemannian case, and negative in the Lorentzian case.

- If $M = \mathbb{M}^2(\kappa)$, the simply-connected surface with constant curvature κ, and τ is constant, we get the BCV-spaces $\mathbb{E}(\kappa, \tau)$ and $\mathbb{L}(\kappa, \tau)$.
- If $\tau = 0$ we get the product manifolds $M \times \mathbb{R}$ and $M \times \mathbb{R}_1$.

Further classification

Let M be a Riemannian surface and $\tau \in \mathcal{C}^\infty(M)$. Then there exists a Killing submersion $\pi : \mathbb{E} \to M$ with bundle curvature τ. It is unique provided that $\mathbb{E}$ is simply-connected.
Such a manifold $\mathbb{E}$ will be denoted by $\mathbb{E}(M, \tau)$ in the Riemannian case, whereas in the Lorentzian case it will be denoted by $\mathbb{L}(M, \tau)$.

- If M is not compact, then π admits a global section $\rightsquigarrow$ quotient of an example in disk-case.
- If M is compact, then there exists a global section $\Leftrightarrow \int_M \tau = 0$.

The trivial submersion $\mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$ and the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ illustrate this last property. Different compact 3-manifolds appear depending on the genus of M.

Vertical graphs

A vertical graph in a Killing submersion $\pi : \mathbb{E} \to M$ is a smooth section of π. Assuming (locally) the model given by $(\star)$, a regular vertical graph Σ_u is the image of

$$(x, y) \mapsto (x, y, u(x, y)),$$

for some open subset $D \subset M$ and $u \in \mathcal{C}^\infty(D)$.
We will introduce the following notation:

Riemannian case	Lorentzian case
$\alpha = u_x + y\,\mathbf{C}_{\lambda,\tau}$	$\widetilde{\alpha} = u_x - y\,\mathbf{C}_{\lambda,\tau}$
$\beta = u_y - x\,\mathbf{C}_{\lambda,\tau}$	$\widetilde{\beta} = u_y + x\,\mathbf{C}_{\lambda,\tau}$
$\omega = \sqrt{1 + \frac{\alpha^2+\beta^2}{\lambda^2}}$	$\widetilde{\omega} = \sqrt{1 - \frac{\alpha^2+\beta^2}{\lambda^2}}$
$Gu = \frac{\alpha\,\partial_x + \beta\,\partial_y}{\lambda^2}$	$Gu = \frac{\alpha\,\partial_x + \beta\,\partial_y}{\lambda^2}$

The vector field $Gu \in \mathfrak{X}(D)$ plays the role of a generalized gradient. The mean curvature of Σ_u admits the following divergence-type expression:

$$H = \frac{1}{2}\mathrm{div}_M\left(\frac{Gu}{\sqrt{1 \pm \|Gu\|_M^2}}\right).$$

Twin correspondence [1]

Theorem. Let M be a Riemannian surface, $H, \tau \in \mathcal{C}^\infty(M)$, and $D \subset M$ open and simply-connected. There is a correspondence between:

1. Graphs with prescribed mean curvature H in $\mathbb{E}(M, \tau)$ over D.
2. Spacelike graphs with prescribed mean curvature τ in $\mathbb{L}(M, H)$ over D.

Some remarks about this correspondence:

- The map $(x, y, u(x, y)) \mapsto (x, y, v(x, y))$ is conformal between two twin surfaces Σ_u and Σ_v, so the conformal structure is preserved.
- It is unique up to an additive constant (i.e., up to a translation in the direction of ξ).
- Using the notation above in the model $(\star)$, it can be computed (locally) explicitly via the twin relations:
  $$\widetilde{\alpha} = \frac{-\beta}{\omega}, \qquad \widetilde{\beta} = \frac{\alpha}{\omega}, \qquad \widetilde{\omega} = \frac{1}{\omega}.$$

The proof relies on a clever use of Poincaré's Lemma and the fact that both H and τ admit divergence-type expressions. Moreover, it represents a natural and significant generalization of the following previous cases:

- Minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{L}^3$ (Calabi, 1970).
- Minimal surfaces in $M \times \mathbb{R}$ and maximal surfaces in $M \times \mathbb{R}_1$ (Albujer-Alías, 2009).
- CMC H surfaces in $\mathbb{E}(\kappa, \tau)$ and CMC τ surfaces in $\mathbb{L}(\kappa, H)$ (Lee, 2011).

Applications

1. Prescribed mean curvature equations. Let us consider the following problems:

- Given $H \in \mathcal{C}^\infty(\mathbb{R}^2)$, is there $u \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that the graph $(x, y) \mapsto (x, y, u(x, y))$ has mean curvature $H(x, y)$ in $\mathbb{R}^3$?

  Several obstructions appear, as the fact that H cannot be bigger than R in a ball of radius $\frac{1}{R}$ for any $R > 0$ (Heinz condition).

  By twin correspondence it is equivalent to the Bernstein problem in $\mathbb{L}(\mathbb{R}^2, H)$, i.e., the existence of entire maximal graphs in $\mathbb{L}(\mathbb{R}^2, H)$.

- Lorentzian counterpart: Given $H \in \mathcal{C}^\infty(\mathbb{R}^2)$, is there $u \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that the graph $(x, y) \mapsto (x, y, u(x, y))$ is spacelike and has mean curvature $H(x, y)$ in $\mathbb{L}^3$?

  Similarly, there is a twin relation with entire minimal graphs in $\mathbb{E}(\mathbb{R}^2, H)$, but in this case we do not have a Heinz condition.

  If H is a radial function, the equation $z = 0$ defines an entire minimal graph in $\mathbb{E}(\mathbb{R}^2, H)$, which gives an easy way of proving that there exist arbitrary prescribed radial mean curvature surfaces in $\mathbb{L}^3$.

2. Complete spacelike surfaces. Let M be non-compact simply-connected. The Cheeger constant of M is defined as

$$\mathrm{Ch}(M) = \inf\left\{\frac{\mathrm{Length}(\partial D)}{\mathrm{Area}(D)} : D \subset M \text{ regular}\right\}.$$

A classical application of the divergence theorem yields that, given $H \in \mathcal{C}^\infty(M)$ such that $\inf_M |H| > \frac{1}{2}\mathrm{Ch}(M)$, the space $\mathbb{E}(M, \tau)$ does not admit entire graphs with mean curvature H.

Since this argument is independent of τ, we can produce a twin result independent of H in the Lorentzian setting:

Theorem. Given $\tau \in \mathcal{C}^\infty(M)$ such that $\inf_M |\tau| > \frac{1}{2}\mathrm{Ch}(M)$, the spacetime $\mathbb{L}(M, \tau)$ does not admit complete spacelike surfaces.

In $\mathbb{L}(M, \tau)$, complete spacelike surfaces are always entire vertical graphs, which gives the key ingredient for this generalization.

It is well-known that $\mathrm{Ch}(\mathbb{R}^2) = 0$, so we get that any Lorentzian Killing submersion over $\mathbb{R}^2$ with bundle curvature bounded away from zero does not admit complete spacelike surfaces. This is the case of $\mathrm{Nil}_3^1(\frac{1}{2}) = \mathbb{L}(\mathbb{R}^2, \frac{1}{2})$.

Lorentzian manifolds not admitting complete spacelike surfaces are not *distinguishable*, in the sense of causality. Nevertheless, many of these spaces are not even causal, since they contain closed timelike curves.

References

[1] H. Lee, J.M. Manzano. Generalized Calabi's correspondence and complete spacelike surfaces. Preprint, 2013 [arXiv:1301.7241].

[2] J.M. Manzano. On the classification of Killing submersions and their isometries. *Pac. J. Math.*, **270** (2014), no. 2, 367–392.

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