Conjugate Plateau Constructions

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Abstract

We construct non-zero constant mean curvature H surfaces in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ by using suitable conjugate Plateau constructions. The resulting surfaces are complete, have bounded height and are invariant under a discrete group of horizontal translations. In $\mathbb{S}^2 \times \mathbb{R}$ (for any $H > 0$) or $\mathbb{H}^2 \times \mathbb{R}$ (for $H > 1/2$), a 1-parameter family of unduloid-type surfaces is obtained, some of which are shown to be compact in $\mathbb{S}^2 \times \mathbb{R}$. Finally, in the case of $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$, the constructed examples have the symmetries of a tessellation of $\mathbb{H}^2$ by regular polygons.

Daniel correspondence

Let $\mathbb{E} = \mathbb{E}(\kappa,\tau)$ and $\mathbb{E}_* = \mathbb{E}(\kappa_*,\tau_*)$ two homogeneous spaces, and consider $H, H_*, \theta \in \mathbb{R}$ such that

$$\kappa - 4\tau^2 = \kappa_* - 4\tau_*^2,$$

$$\tau + iH = e^{i\theta}(\tau_* + iH_*).$$

Given a simply-connected Riemannian surface Σ, there is a correspondence between H-immersions $\phi : \Sigma \to \mathbb{E}$ and H_*-immersions $\phi_* : \Sigma \to \mathbb{E}_*$, and they are called *sister surfaces*.

An interesting case ($H \in \mathbb{R}$, $\epsilon \in \{-1,0,1\}$)

$$\begin{array}{ccc} \Sigma \looparrowright \mathbb{E}(4H^2+\epsilon, H) & \longleftrightarrow & \Sigma \looparrowright \mathbb{M}^2(\epsilon) \times \mathbb{R} \\ \text{minimal} & [\theta = \frac{\pi}{2}] & \text{CMC-}H \end{array}$$

Correspondence between curves. Given a smooth curve $\alpha : [a,b] \to \Sigma$:

$$\begin{array}{lll} \phi(\alpha) \text{ horiz. geodesic} & \to & \phi_*(\alpha) \subset \text{vert. plane} \\ \phi(\alpha) \text{ vert. geodesic} & \to & \phi_*(\alpha) \subset \text{horiz. plane} \end{array}$$

Moreover, ϕ_* meets the plane orthogonally.

Correspondence between symmetries.

$\phi(\Sigma)$ invariant by reflection respect to ⟶ $\phi^*(\Sigma)$ invariant by reflection respect to

$$\begin{array}{lll} \text{horizontal geodesic} & \to & \text{vertical plane} \\ \text{vertical geodesic} & \to & \text{horizontal plane} \end{array}$$

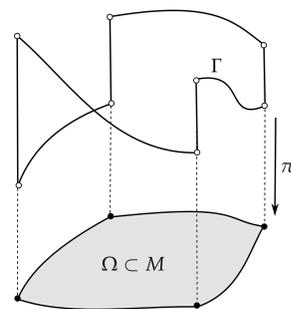Moreover, ϕ^* meets the plane of reflection at $\phi^*(\alpha)$.

References

[1] José M. Manzano, F. Torralbo. New examples of contant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. Preprint available at arXiv:1104.1259[math.DG].

The Plateau problem

Let $\pi : \mathbb{E} \to M$ be a Killing submersion whose fibers have infinite length and $\Gamma \subset \mathbb{E}$ a Nitsche contour over a convex domain $\Omega \subset M$.

- The Plateau problem has a unique solution $\Sigma \subset \pi^{-1}(\overline{\Omega})$ with boundary Γ.
- The interior of Σ is a graph over Ω.
- Σ inherits the symmetries of Γ.

Extension of the pieces

Given a H-surface Σ in $\mathbb{E}(\kappa,\tau)$:

- If $\Gamma \subset \partial\Sigma$ is a horizontal or vertical geodesic, then Σ extends across Γ by reflection w.r.t. Γ.
- If $\Gamma \subset \partial\Sigma$ is contained in a vertical or horizontal plane Π orthogonal to Σ, then Σ extends across Γ by a reflection w.r.t. Π.

Minimal case. If $\partial\Sigma$ is made out of vertical and horizontal geodesics meeting at angles of $\frac{\pi}{k}$, $k \in \mathbb{N}$, then Σ extends to a complete minimal surface.

CMC case. If $\partial\Sigma$ is made out of curves lying on vertical and horizontal planes (which are orthogonal to Σ) meeting at angles of $\frac{\pi}{k}$, $k \in \mathbb{N}$, then Σ extends to a complete H-surface.

Horizontal unduloids

**The minimal piece in $\mathbb{S}_b^3(4H^2 + \epsilon, H)$.** Given $\lambda \in [0, \frac{\pi}{2}]$, consider the Nitsche contour:
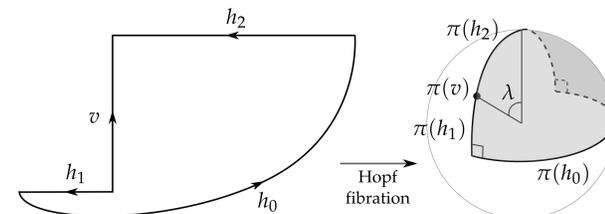
and Σ_λ, the solution of the Plateau problem.

- $\{\text{int}(\Sigma_\lambda)\}_\lambda$ foliates an open set in $\mathbb{S}_b^3(4H^2+\epsilon, H)$, bounded by the minimal sphere $\Sigma_{\pi/2}$, the helicoid Σ_0 and a Clifford torus.
- The angle function at the boundary is strictly monotonic with respect to λ.
- The angle function does not attain the values 0 or -1 at the interior of Σ_λ.

The sister piece in $\mathbb{M}^2(\epsilon) \times \mathbb{R}$. Consider the sister H-surface Σ_λ^* and extend to $\widetilde{\Sigma}_\lambda^*$.
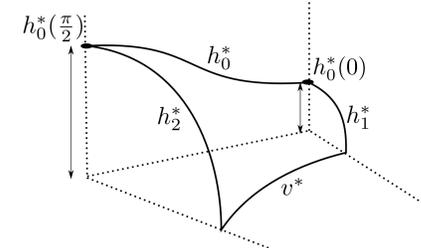
- $\widetilde{\Sigma}_\lambda^*$ gives a continuous deformation between the horizontal cylinder $\widetilde{\Sigma}_0^*$ and the sphere $\widetilde{\Sigma}_{\pi/2}^*$.
- Σ_λ^* attains its maximum height at $h_0^*(\frac{\pi}{2})$.
- We obtain many compact examples in $\mathbb{S}^2 \times \mathbb{R}$.

Spherical helicoids

The minimal immersions $\Phi_c : \mathbb{R}^2 \to \mathbb{S}_b^3(4H^2 + \epsilon, H)$,
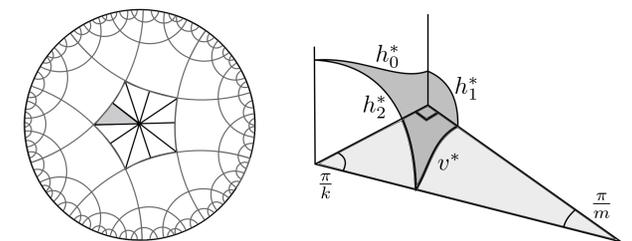
$$\Phi_c(x,y) = (\cos(x)e^{icy}, \sin(x)e^{iy}),$$

have the following sister H-surfaces in $\mathbb{M}^2(\epsilon) \times \mathbb{R}$:

$$\begin{array}{lcl} c = 1 & \rightsquigarrow & \text{vertical cylinder} \\ 0 < c < 1 & \rightsquigarrow & \text{vertical unduloids} \\ c = 0 & \rightsquigarrow & \text{sphere} \\ -1 < c < 0 & \rightsquigarrow & \text{vertical nodoids} \\ c = -1 & \rightsquigarrow & \text{horizontal cylinder} \end{array}$$

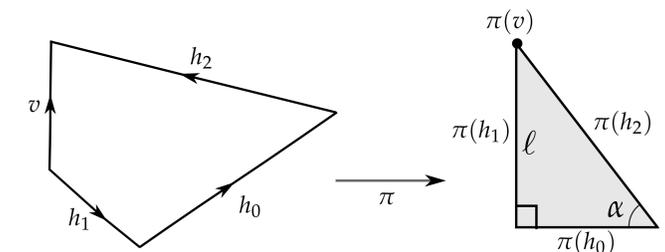## 1/2-surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Regular tessllations of $\mathbb{H}^2 \times \mathbb{R}$. Given $m, k \in \mathbb{N}$, there exists a tessellation of m-gons of $\mathbb{H}^2 \times \mathbb{R}$, k of them meeting at each vertex, if and only if

$$\frac{1}{m} + \frac{1}{k} < \frac{1}{2}.$$

We want our piece to fit the shaded triangle.

The minimal piece in $\text{Nil}_3 = \mathbb{E}(0, \frac{1}{2})$. Given $\ell > 0$ and $0 < \alpha < \frac{\pi}{2}$, consider the Nitsche contour:

and $\Sigma_{\ell,\alpha}$, the solution of the Plateau problem.

Key property. For any fixed α, the integral of the angle function of $\Sigma_{\ell,\alpha}$ along h_0 diverges when $\ell \to \infty$, so the length of $\pi(h_0^*)$ also diverges.

The sister surface. Fix $\alpha = \frac{\pi}{k}$ and choose ℓ so $\pi(h_0^*)$ has the desired length by the key property. Then, hyperbolic geometry makes everything fit.

Quotients of $\mathbb{H}^2 \times \mathbb{R}$

Regular gluing patterns. By making an identification between the sides of a regular tessellation of $\mathbb{H}^2$ so all vertices glue together, the quotient surface M is compact and has negative Euler characteristic.

1/2-surfaces in $M \times \mathbb{R}$. The complete surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ produced as above gives a compact surface $\widehat{\Sigma}$ in $M \times \mathbb{R}$ after a regular gluing pattern. Moreover:

- $\widehat{\Sigma}$ is orientable $\Leftrightarrow$ Σ is orientable.
- If Σ has genus g, then $\widehat{\Sigma}$ has genus $2g$.