Surfaces in Riemannian and Lorentzian 3-manifolds admitting a Killing vector field

Ana M. Lerma
Universidad de Jaén
alerma@ujaen.es

José M. Manzano
King's College London
manzanoprego@gmail.com

Abstract

Riemannian and Lorentzian Killing submersions over a given surface M can be classified in terms of two geometric functions in M, namely the *bundle curvature* τ and the *Killing length* μ. These two functions also yield restrictions to the topology of the total space of the submersion. If the base is simply connected, τ and μ can be prescribed arbitrarily giving rise to a unique Killing submersion structure, but uniqueness fails if simple connectedness is dropped. **In the first part**, we will solve the generalized Bernstein problem in a Riemannian Killing submersion over a compact base, giving some applications to the classification of compact stable surfaces with constant mean curvature, and characterizing the existence of minimal spheres in these spaces [2]. **In the second part**, we will show the existence of a constant $\mathrm{Ch}(M,\mu)$ such that the total space of any Lorentzian Killing submersion over M with Killing length μ and bundle curvature τ satisfying $\inf_M|\tau| > \frac{1}{2}\mathrm{Ch}(M,\mu)$ does not admit complete spacelike surfaces. This is based in a joint work of the second author and H. Lee [1].

Riemannian and Lorentzian Killing submersions

Killing submersions. A Riemannian submersion $\pi:\mathbb{E}\to M$ is called *Killing* if its fibers are the integral curves of a Killing vector field ξ in $\mathbb{E}$ without zeroes.

In the sequel $\mathbb{E}$ and M will be a 3-manifold and a surface, respectively. They will be asummed connected and orientable. If ξ is timelike then π is said **Lorentzian**.

Classification ingredients:
- The **base surface** M.
- The **Killing length** is the positive function $\mu=\|\xi\|$.
- The **bundle curvature** is the function given by
$$\tau(p)=\frac{1}{\mu(p)}\langle\overline{\nabla}_{e_1}\xi,e_2\rangle,$$
where $\{e_1,e_2\}$ is an orthonormal frame of $\ker(\mathrm{d}\pi_p)^\perp$.

The functions τ and μ are constant along fibres, so they induce smooth functions $\tau,\mu\in C^\infty(M)$. Then it makes sense to classify Killing submersions in terms of (M,τ,μ).

Classification criterion: Two Killing submersions $\pi_i:\mathbb{E}_i\to M_i$, $i\in\{1,2\}$, are **isomorphic** if there are isometries $f:\mathbb{E}_1\to\mathbb{E}_2$, $h:M_1\to M_2$ such that $\pi_2\circ f=h\circ\pi_1$.

Simply-connected case. Let M be a simply-connected surface and let $\tau,\mu\in C^\infty(M)$, $\mu>0$. There exists a unique Killing submersion $\pi:\mathbb{E}\to M$ such that
- π has prescribed bundle curvature τ and Killing length μ.
- $\mathbb{E}$ is simply-connected.

Let M be $\Omega\subset\mathbb{R}^2$, a star-shapped domain w.r.t. the origin, endowed with the conformal metric $\lambda^2(\mathrm{d}x^2+\mathrm{d}y^2)$, and consider

$$\mathbb{E}(M,\tau,\mu)=(\Omega\times\mathbb{R},\lambda^2(\mathrm{d}x^2+\mathrm{d}y^2)+\mu^2(\mathrm{d}z+\eta(y\,\mathrm{d}x-x\,\mathrm{d}y))^2,$$
$$\mathbb{L}(M,\tau,\mu)=(\Omega\times\mathbb{R},\lambda^2(\mathrm{d}x^2+\mathrm{d}y^2)-\mu^2(\mathrm{d}z-\eta(y\,\mathrm{d}x-x\,\mathrm{d}y))^2,$$

$$\eta(x,y)=\int_0^1\frac{2s\,\tau(xs,ys)\lambda(xs,ys)^2}{\mu(xs,ys)}\,\mathrm{d}s.$$

The unique Killing submersions over M in the above conditions are, up to isomorphism, the projections over the first factor $\pi:\mathbb{E}(M,\tau,\mu)\to M$ and $\pi:\mathbb{L}(M,\tau,\mu)\to M$.

Non-simply-connected case. Given a Killing submersion $\pi:\mathbb{E}\to M$, let $\widetilde{\mathbb{E}}$ and $\widetilde{M}$ be the univ. covers of $\mathbb{E}$ and M.
- There exists a Killing submersion structure $\widetilde{\pi}:\widetilde{\mathbb{E}}\to\widetilde{M}$.
- There is a group of Killing isometries $G\leq\mathrm{Iso}(\widetilde{\mathbb{E}})$ acting properly discontinuously on $\widetilde{\mathbb{E}}$ such that $\mathbb{E}=\widetilde{\mathbb{E}}/G$.

Killing graphs

Killing graphs. Let $\pi:\mathbb{E}\to M$ be a Killing submersion. A *Killing graph* over $\Omega\subset M$ is a transversal spacelike section $F:\Omega\to\pi^{-1}(\Omega)$. If $\Omega=M$, the graph is said *entire*.

Given an initial smooth section $F_0:\Omega\to\mathbb{E}$, a (smooth) Killing graph over Ω can be parametrized in terms of $u\in C^\infty(\Omega)$ as
$$F_u(p)=\phi_{u(p)}(F_0(p)),$$
where ϕ_t stands for the vertical translation of length t.

Let $\epsilon=1$ in the Riemannian case, $\epsilon=-1$ otherwise.

The mean curvature of the surface F_u is given by
$$2H\mu=\mathrm{div}\left(\frac{\mu Gu}{\sqrt{\mu^{-2}+\epsilon\|Gu\|^2}}\right),$$
where $Gu=\nabla u-Z$ and $Z\in\mathfrak{X}(M)$ does not depend upon u, satisfying the divergence equation $\mathrm{div}(JZ)=\frac{-2\varepsilon\tau}{\mu}$.

Existence of global sections. A Killing submersion $\pi:\mathbb{E}\to M$ admits a global section if and only if either M is noncompact or M is compact and $\int_M\frac{\tau}{\mu}=0$.

Entire minimal graphs

Theorem. Let $\pi:\mathbb{E}\to M$ be a **Riemannian** Killing submersion, such that M is compact. Let τ and μ be its bundle curvature and Killing length, respectively.
- If $\int_M\frac{\tau}{\mu}=0$, then there is an entire minimal graph $\Sigma\subset\mathbb{E}$.
- If $\int_M\frac{\tau}{\mu}\neq0$, then π does not admit global sections.

The entire minimal graph Σ is diffeomorphic to M, and admits the following characterizations (up to translations):

- **Bernstein problem.** Σ is the only entire minimal graph in $\mathbb{E}$ whose mean curvature does not change sign.
- **Plateau problem.** Σ is area-minimizing, i.e., it is a global minimum of the area functional
$$\mathcal{A}(u)=\int_M\sqrt{\mu^{-2}+\|Gu\|^2}$$
among all Lipschitz functions in M.
Note that $\mathcal{A}(u)=\int_M\sqrt{f^2+\|\nabla u-Z\|^2}$, and $f\in C^\infty(M)$ and $Z\in\mathfrak{X}(M)$ can be prescribed in terms of τ and μ.
- **Hopf problem.** If the fibers of π have infinite length, then Σ is the only compact minimal surface in $\mathbb{E}$.

Sketch of the proof.
- If M is not a sphere, the result follows from the existence of minimizers of the area in a isotopy class of surfaces (Meeks-Simon-Yau, 1982).
- If M is a sphere, then we can apply the Calabi-type correspondence and the existence of prescribed mean curvature spacelike graphs in $\mathbb{L}(M,0,\frac{1}{\mu})$ (Gerhardt, 1983).

Stable CMC surfaces

Stable CMC H surfaces are second-order minima of $\mathcal{J}=\mathrm{Area}-2H\cdot\mathrm{Vol}$, for *all* compactly supported variations.

If Σ is a compact CMC-H surface in the total space of a **Riemannian** Killing submersion $\pi:\mathbb{E}\to M$, and Σ is oriented by a unit normal N, then the *angle function* $\nu=\langle N,\xi\rangle$ lies in one of these two situations:

- $\nu\equiv0\Rightarrow\Sigma$ tangent to the vertical direction.
Then $\Sigma=\pi^{-1}(\Gamma)$ is the *vertical cylinder* over a closed curve $\Gamma\subset M$, so the fibers of π are compact and $\Sigma\cong\mathbb{S}^1\times\mathbb{S}^1$.
- ν **never vanishes** $\Rightarrow\Sigma$ transversal to the Killing direction.
Then M is compact, and Σ is an entire minimal graph. In particular, $\int_M\frac{\tau}{\mu}=0$.

These are the only two scenarios such that $\mathbb{E}$ admits compact orientable stable CMC surface Σ.
– If Σ is not a torus, then Σ is an entire minimal graph.
– If $H\neq0$, then Σ is a vertical cylinder

Minimal spheres

The existence of entire minimal graphs and the existence of index-1 minimal spheres in 3-spheres (Simon, 1985), yield:

Theorem. Let $\pi:\mathbb{E}\to M$ be a **Riemannian** Killing submersion. Then $\mathbb{E}$ admits an immersed minimal sphere if and only if M is diffeomorphic to a sphere.

Such a minimal sphere is an entire minimal graph if $\int_M\frac{\tau}{\mu}=0$, but embeddedness may fail if $\int_M\frac{\tau}{\mu}\neq0$ and $\mathbb{E}$ is not simply-connected. The only stable ones are the entire minimal graphs.

Uniqueness of minimal spheres is not guaranteed in general.

Calabi-type correspondence

Theorem. Let M be a simply-connected surface, and consider arbitrary functions $H,\tau,\mu\in C^\infty(M)$, $\mu>0$. There is a correspondence between the following two families:
- Entire graphs in $\mathbb{E}(M,\tau,\mu)$ with prescribed mean curvature H.
- Entire spacelike graphs in $\mathbb{L}(M,H,\frac{1}{\mu})$ with prescribed mean curvature τ.

This correspondence preserves the conformal type, and it is one-to-one up to vertical translations. Note that both families are empty if M is a sphere and $\int_M\frac{\tau}{\mu}\neq0$.

This result generalizes the classical correspondence between minimal graphs in $\mathbb{R}^3$ and spacelike maximal graphs in $\mathbb{L}^3$ (Calabi, 1970), as well as others intermediate correspondences (Albujer-Alías, 2009), (Lee, 2011).

The proof relies in divergence-type expressions for the mean curvature and the bundle curvature in a Killing submersion, together with a clever use of Poincaré's Lemma.

For instance we get a correspondence between:
– Entire graphs with prescribed mean curvature in $\mathbb{R}^3$.
– Entire spacelike minimal graphs in the generalized Lorentzian Heisenberg space $\mathbb{L}(\mathbb{R}^2,\tau,1)$.

Cheeger constant

Let M be a non-compact Riemannian surface. We define the Cheeger constant in M with density $\mu\in C^\infty(M)$ (positive) as
$$\mathrm{Ch}(M,\mu)=\inf\left\{\frac{\int_{\partial\Omega}\mu}{\int_\Omega\mu}:\Omega\subset\subset M\text{ regular}\right\}\geq0.$$

Adapting a standard computation (Heinz, 1955) to the Riemannian Killing-submersion setting, we get that

If $H\in C^\infty(M)$ is such that $\inf_M|H|>\frac{1}{2}\mathrm{Ch}(M,\mu)$, then a Riem. Killing submersion over M with Killing length μ does not admit entire graphs with prescribed mean curvature H.

Now we get the dual of this result, using the fact that entire spacelike graphs are the only complete spacelike surfaces in the Lorentzian setting:

If $\tau\in C^\infty(M)$ is such that $\inf_M|\tau|>\frac{1}{2}\mathrm{Ch}(M,\mu)$, then a Lorentzian Killing submersion over M with Killing length μ does not admit complete spacelike surfaces.

Note that the Riemannian result does not depend on τ, and the Lorentzian counterpart does not depend on H. In M is simply-connected, both results are equivalent via the correspondence.

To prove that this non-existence result is sharp, let us consider $\mathbb{E}(\kappa,\tau)$-spaces, with $\kappa\leq0$, which are Killing sumersions over $\mathbb{M}^2(\kappa)$ with constant bundle curvature τ and constant Killing length. It is easy to show that $\mathrm{Ch}(\mathbb{M}^2(\kappa),1)=\sqrt{-\kappa}$, and also:

- If $H\geq c>\frac{1}{2}\sqrt{-\kappa}$, for some constant c then there do not exist entire graphs with mean curvature H in $\mathbb{E}(\kappa,\tau)$.
- If $H\leq\frac{1}{2}\sqrt{-\kappa}$ is constant, then there exist entire graphs in $\mathbb{E}(\kappa,\tau)$ with CMC H (e.g., rotationally-invariant).

Open questions

1. We conjecture that any Riemannian Killing submersion that admits a global section, also admits a global *minimal* section. This is proved in [2] under the assumption that the base M is compact.

   We also conjecture that any Lorentzian Killing submersion that admits a global spacelike section, also admits a global spacelike section of prescribed mean curvature. If the base is simply-connected, both conjectures are dual.

2. Non-existence of compact orientable stable CMC surfaces can be regarded as an estimate of the first eigenvalue of the stability operator for such a surface. Nonetheless, getting a sharp estimate for this eigenvalue in a Killing submersion in terms of (M,τ,μ) is far from being solved, not even in the much simpler case of $\mathbb{E}(\kappa,\tau)$-spaces.

References

[1] Lee, H., Manzano, J.M., Generalized Calabi's correspondence and complete spacelike surfaces. Available at arXiv:1301.7241.
[2] Lerma, A.M., Manzano, J.M., Compact stable surfaces with constant mean curvature in Killing submersions. Available at arXiv:1604.00542.

Partially supported by MEC-Feder research project MTM2014-52368-P. The second author was also supported by EPSRC grant EP/M024512/1.