Area estimates for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$-spaces

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Abstract

We obtain area growth estimates for constant mean curvature graphs in $\mathbb{E}(\kappa,\tau)$-spaces with $\kappa\leq 0$ by finding sharp upper bounds for the volume of metric balls in $\mathbb{E}(\kappa,\tau)$. We focus on complete graphs and graphs with zero boundary values. For instance, we prove that entire graphs in $\mathbb{E}(\kappa,\tau)$ with critical mean curvature have at most cubic intrinsic area growth. We also obtain sharp upper bounds for the extrinsic area growth of graphs with zero boundary values, and study distinguished examples in detail such as invariant surfaces, k-noids and ideal Scherk graphs. Finally we give a relation between height and area growth of minimal graphs in the Heisenberg space ($\kappa=0$), and prove a Collin-Krust type estimate for such minimal graphs.

Estimating the area of surfaces

The space $\mathbb{E}(\kappa,\tau)$ is characterized by admitting a Killing submersion π structure over $\mathbb{M}^2(\kappa)$ with constant bundle curvature τ. There are three natural ways of estimating the area of a **properly immersed** surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$:

IAG (Intrinsic area growth)
- Growth of $R\mapsto \mathrm{Area}(B_R^\Sigma(p_0))$
- $B_R^\Sigma(p_0)$ = intrinsic ball of radius R centered at $p_0\in\Sigma$.

EAG (Extrinsic area growth)
- Growth of $R\mapsto \mathrm{Area}(\Sigma\cap B_R(p_0))$
- $B_R(p_0)$ = extrinsic ball of radius R centered at $p_0\in\mathbb{E}(\kappa,\tau)$.

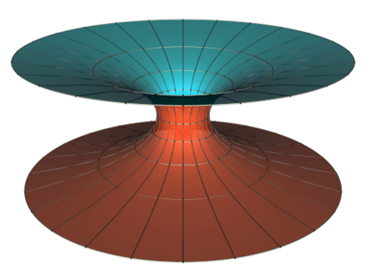
CAG (Cylindrical area growth)
- Growth of $R\mapsto \mathrm{Area}(\Sigma\cap C_R(p_0))$
- $C_R(p_0)=\pi^{-1}(D_R(p_0))$, where $D_R(p_0)$ is a disk centered at $p_0\in\mathbb{M}^2(\kappa)$.

Immediate properties
- These definitions depend neither on the choice p_0 nor on modifying Σ in a compact set.
- Since $B_R^\Sigma(p_0)\subset\Sigma\cap B_R(p_0)\subset\Sigma\cap C_R(p_0)$ holds, we easily get **IAG** $\leq$ **EAG** $\leq$ **CAG**.

Classical results (Σ complete, $\partial\Sigma=\emptyset$)
- (Hartman, 64) $K^-\in L^1(\Sigma)\implies$ **IAG** $\leq$ quadratic
- (Cheng-Yau, 75) **IAG** $\leq$ quadratic $\implies$ parabolicity
- (Li, 97) **IAG** $\leq$ quadratic and $K\leq 0\implies K\in L^1(\Sigma)$

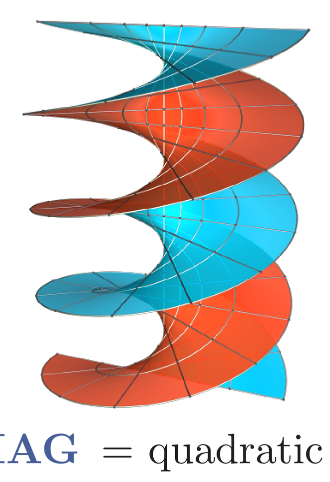
Some well-known minimal examples in $\mathbb{R}^3$ (images by Mathias Weber)

CATENOID



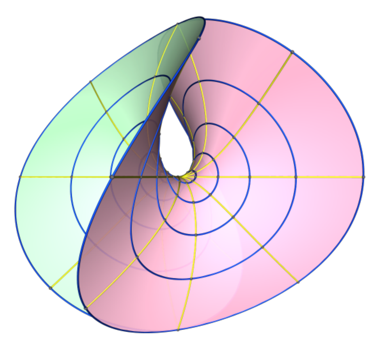
IAG = quadratic
EAG = quadratic

HELICOID



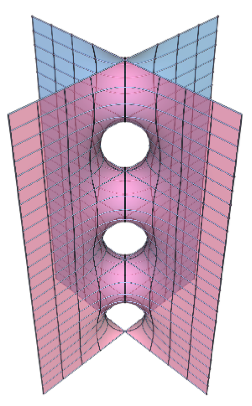
IAG = quadratic
EAG = cubic

ENNEPER



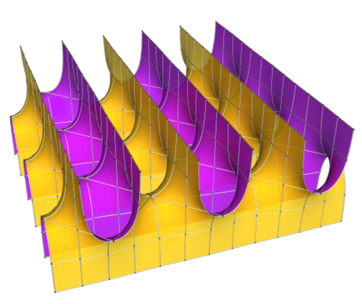
IAG = quadratic
EAG = cubic

S.P. SCHERK



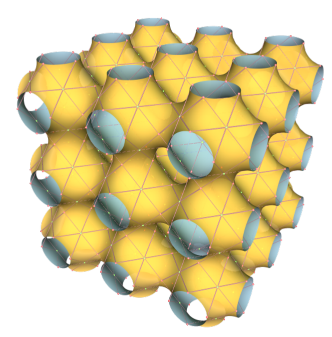
IAG = quadratic
EAG = quadratic

D.P. SCHERK



IAG = quadratic
EAG = cubic

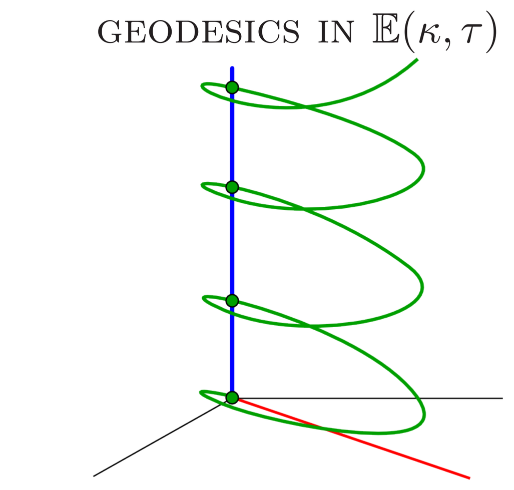
TRIPLY PERIODIC



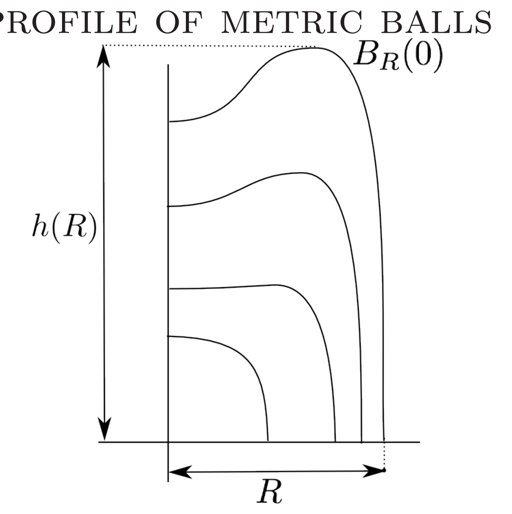
IAG = cubic
EAG = cubic

Metric balls in $\mathbb{E}(\kappa,\tau)$

GEODESICS IN $\mathbb{E}(\kappa,\tau)$



PROFILE OF METRIC BALLS



Geodesic	Projection
vertical	point
horizontal	$\kappa_g=0$
helicoidal	$\kappa_g=\text{const.}$

$$h(R)\sim\begin{cases}R^2 & \kappa=0,\tau\neq0\\ R & \kappa=\tau=0\\ R & \kappa\neq0\end{cases}$$

- If a surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$ has an **embedded metric neighborhood** of uniform radius, then the **EAG** is at most as $R\mapsto\mathrm{Area}(B_R(p))$.
  - In $\mathrm{Nil}_3(\tau)$, $\mathrm{Area}(B_R(p))$ grows as R^4.
  - If $\kappa<0$, $\mathrm{Area}(B_R(p))$ grows exponentially.
- No lower bound or sharp monotonicity formula is hitherto known except for minimal surfaces in $\mathbb{R}^3$.

Cylindrical area growth

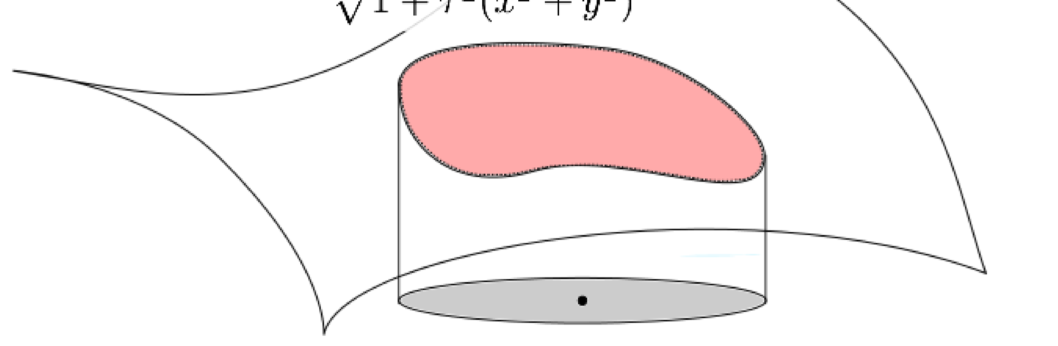
The **vertical graph** of a function u defined in $\Omega\subset\mathbb{M}^2(\kappa)$ is the surface Σ_u parameterized by $F_u(x,y)=(x,y,u(x,y))$. Its mean curvature H admits a divergence equation

$$H=\frac{1}{2}\mathrm{div}\left(\frac{\nabla u+Z}{\sqrt{1+\|\nabla u+Z\|^2}}\right),$$

where div, ∇ and $\|\cdot\|$ are computed in M. Here Z is a vector field in M not depending upon u.
If $D_R(0)\subset\Omega$, divergence theorem yields

$$\mathrm{Area}(\Sigma_0)=\int_{\Sigma_u\cap C_R(0)}\langle N,N_0\rangle\leq\mathrm{Area}(\Sigma_u\cap C_R(0)).$$

$$N_0=\frac{-\tau x\partial_x+\tau y\partial_y+(1+\tau^2(x^2+y^2))\partial_z}{\sqrt{1+\tau^2(x^2+y^2)}}$$



Hence **CAG**$(\Sigma_u)\geq$ **CAG**(Σ_0), with equality $\Leftrightarrow u\equiv 0$.

Extrinsic area growth

Let $\Sigma_u\subset\mathbb{E}(\kappa,\tau)$ be a minimal graph over a domain Ω. Assume that $\partial\Omega$ is piecewise smooth and consists of

1. regular arcs where u has continuous limit values, and
2. regular arcs where u has $\pm\infty$ limit values.

Let $\Omega(R)=\Omega\cap D_R(0)$ and let $\ell(R)$ be the length of $\{p\in\partial\Omega(R):u(p)\neq0\}$. Then we can bound the **EAG** in terms of the geometry of the base and the height of metric balls:

$$\boxed{\mathrm{Area}(\Sigma_u\cap B_R(0))\leq\int_{\Omega(R)}(1+|Z|)+h(R)\ell(R).}$$

If one of the following conditions holds:

- u extends continuously to $\partial\Omega$ as cero,
- $R\mapsto\ell(R)$ grows as $R\mapsto\mathrm{Length}(D_R(0))$;

then we obtain the following bound for **EAG**:

- **EAG**$(\Sigma_u)\leq$ quadratic in $\mathbb{R}^3$,
- **EAG**$(\Sigma_u)\leq$ cubic in $\mathrm{Nil}_3(\tau)$,
- **EAG**$(\Sigma_u)\leq Re^{R\sqrt{-\kappa}}$ if $\kappa<0$.

Height estimates in Nil_3

Let $\Sigma_u\subset\mathrm{Nil}_3(\tau)$ be an entire minimal graph. Then

(a) **IAG**$(\Sigma)\leq$ **EAG**$(\Sigma)\leq$ cubic.

(b) **CAG**$(\Sigma)\geq$ cubic.

Is it possible to find a condition such that **EAG** = **CAG**?

> If $|u|\leq M(1+x^2+y^2)^\beta$ for some $M>0$ and $\beta\geq1$, then Σ_u has at most **EAG** of order $\frac{3}{\beta}$.

By using Lee's twin correspondence and some gradient inequalities that follow from Cheng-Yau and Treibergs, we can prove that Σ_u satisfies the above property for $\beta=\frac{3}{2}$, i.e., any entire minimal graph in Nil_3 has **at most cubic height growth**.

In the other direction we can prove **at least linear height growth** *à la* Collin-Krust: If $\Sigma\subset\mathrm{Nil}_3(\tau)$ is a minimal graph over an unbounded domain Ω with zero boundary values and $M(R)=\sup\{|u(p)|:p\in\Omega(R)\}$, then

$$\liminf_{R\to\infty}\frac{M(R)}{R}>0.$$

$\mathbb{E}(\kappa,\tau)$-spaces

The space $\mathbb{E}(\kappa,\tau)$ is a **homogeneous** 3-manifold that admits a **Riemannian submersion** over $\mathbb{M}^2(\kappa)$ such that the fibers are the integral curves of a **unit Killing vector field**.

	$\kappa<0$	$\kappa=0$	$\kappa>0$
$\tau=0$	$\mathbb{H}^2\times\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{S}^2\times\mathbb{R}$
$\tau\neq0$	$\widetilde{\mathrm{SL}}_2(\mathbb{R})$	Nil_3	Berger $\mathbb{S}^3$

If $\kappa\leq0$ and $\Omega_\kappa=\{(x,y\in\mathbb{R}^2:1+\frac{\kappa}{4}(x^2+y^2)>0)$, we will consider the model $\mathbb{E}(\kappa,\tau)=\Omega_\kappa\times\mathbb{R}$ with the metric

$$\frac{\mathrm{d}x^2+\mathrm{d}y^2}{\left(1+\frac{\kappa}{4}(x^2+y^2)\right)^2}+\left(\mathrm{d}z^2+\frac{\tau(y\mathrm{d}x-x\mathrm{d}y)}{1+\frac{\kappa}{4}(x^2+y^2)}\right)^2.$$

Then ∂_z is Killing and $(x,y,z)\mapsto(x,y)$ is the aforesaid submersion over $\mathbb{M}^2(\kappa)$.

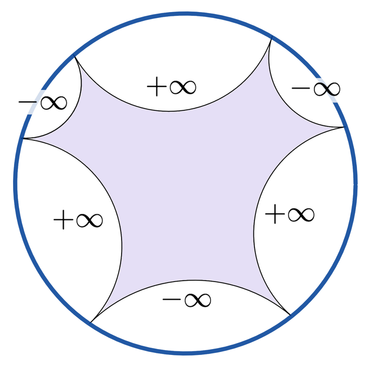
Entire CMC graphs

Entire CMC graphs in $\mathbb{E}(\kappa,\tau)$ with **critical mean curvature** ($4H^2+\kappa=0$, $\kappa\leq0$) were classified in [1] in terms of holomorphic quadratic differentials defined on $\mathbb{C}$ or $\mathbb{D}$, and also by means of Daniel's correspondence. We can bound the area growth of an entire minimal graph $\Sigma_u\subset\mathrm{Nil}_3(\tau)$ in terms of u:

- **Gradient estimate**: $|\nabla u+Z|\leq M(1+x^2+y^2)$
- **Height estimate**: $|u|\leq M(1+x^2+y^2)^{3/2}$
- **Extrinsic area estimate**: quadratic $\leq$ **EAG**$(\Sigma_u)\leq$ cubic
- **Cylindrical area estimate**: cubic $\leq$ **CAG**$(\Sigma_u)\leq$ quartic

> $|u|\leq M(1+x^2+y^2)\implies$ **EAG**(Σ_u) = **CAG**(Σ_u) = cubic.

Ideal Scherk surfaces

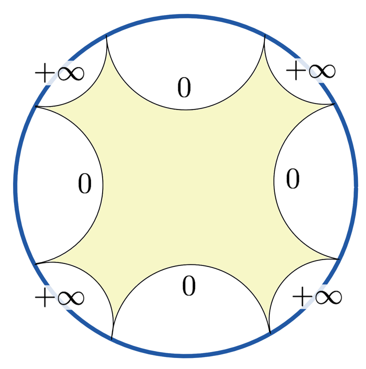


These are **complete graphs** taking alternative values $\pm\infty$ along the $2n$ edges of an ideal polygon in $\mathbb{H}^2(\kappa)$, with **subcritical** CMC ($4H^2+\kappa<0$). They were constructed by Collin-Rosenberg ($\mathbb{H}^2\times\mathbb{R}$) and Folha-Melo ($\widetilde{\mathrm{SL}}_2(\mathbb{R})$, $H=0$), under certain sharp conditions.

- The area of the polygon is $\frac{2(n-1)\pi}{-\kappa-4H^2}$.
- These surfaces are preserved by Daniel's correspondence.

> Ideal Scherk graphs have **IAG** $\leq$ quadratic.

Symmetric CMC k-noids



These are **complete bigraphs** with k ends over some domain of $\mathbb{H}^2(\kappa)$, with **subcritical** CMC ($4H^2+\kappa<0$). They were constructed by Daniel-Hauswirth, Morabito-Rodríguez and Pyo ($\mathbb{H}^2\times\mathbb{R}$, $H=0$), and Plehnert ($\mathbb{H}^2\times\mathbb{R}$, $0<H<\frac{1}{2}$, symmetric).

- In the symmetric case, the surface can be decomposed in $4k$ congruent graphs with boundary.
- Each piece corresponds to a minimal graph in $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ by Daniel's correspondence, where our results apply.

> Symmetric k-noids have **IAG** $\leq$ quadratic.

Open questions

- All known entire minimal graphs in $\mathrm{Nil}_3(\tau)$ have at most quadratic height growth. We conjecture that this holds true for all entire minimal graphs. Were it the case, all entire minimal graphs in $\mathrm{Nil}_3(\tau)$ would have **EAG** = **CAG** = cubic.
- No sharp monotonicity formula is known for a minimal surface $\Sigma\subset\mathrm{Nil}_3(\tau)$. For any known example, the inequality $\mathrm{Area}(\Sigma\cap B_R(p))\geq CR^3$ holds for some constant C.
- It was conjectured by Pérez, Rodríguez and the author that any **complete stable** CMC surface with quadratic **IAG** must be a vertical plane. Here we also conjecture that, apart from vertical planes, any complete stable CMC surface (in particular, an entire minimal graph) has cubic **IAG**.

References

[1] I. Fernández and P. Mira, Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space, *Trans. Amer. Math. Soc.* **361** (2011), 5737–5752.

[2] J. M. Manzano and B. Nelli, Height and area estimates for constant mean curvature graphs in $\mathbb{E}(\kappa,\tau)$-spaces. Preprint available in arXiv:1504.05239.

Research partially supported by EPSRC grant no. EP/M024512/1 and Spanish MEC-Feder research project MTM2014-52368-P.