Conjugate Plateau Constructions

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In 1970, Lawson [Law70] established a correspondence between simply-connected minimal surfaces in a space form $\mathbb{M}^3(\kappa)$, with constant sectional curvature κ, and constant mean curvature H surfaces (H-surfaces in the sequel) in $\mathbb{M}^3(\kappa - H^2)$. He applied this correspondence to obtain doubly-periodic mean curvature one surfaces in $\mathbb{R}^3$. The technique he used to construct such examples is known as the *conjugate Plateau construction* and it has become a fruitful method to obtain constant mean curvature surfaces in space forms. The main steps of this construction are the following:

1. Solve the Plateau problem in a geodesic polygon in $\mathbb{M}^3(\kappa)$.
2. Consider the *conjugate* H-surface in $\mathbb{M}^3(\kappa - H^2)$, whose boundary lies on some planes of symmetry since the initial surface is bounded by geodesic curves.
3. Reflect the resulting surface across its edges to get a complete (not necessarily embedded) H-surface in $\mathbb{M}^3(\kappa - H^2)$.

In the poster, we will discuss all the ingredients necessary to extend the conjugate Plateau construction to the case of the spaces $\mathbb{E}(\kappa,\tau)$, the simply-connected Riemannian homogenous 3-manifolds whose isometry group has dimension 4. Note that these spaces admit a Riemannian submersion $\pi : \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ with constant bundle curvature τ, and the fibers of π are the integral curves of a unit Killing vector field ξ in $\mathbb{E}(\kappa,\tau)$. After doing that, we will apply the conjugate Plateau construction in order to obtain some interesting H-surfaces in the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (cf. [MT11]).

The aforementioned ingredients can be summarized in the following two items:

- The first tool we will need is an analogous to the Lawson correspondence in the $\mathbb{E}(\kappa,\tau)$-setting, a role which is played by the Daniel correspondence. Briefly, given $\mathbb{E} = \mathbb{E}(\kappa,\tau)$ and $\mathbb{E}^* = \mathbb{E}(\kappa^*,\tau^*)$ such that $\kappa - 4\tau^2 = \kappa^* - 4(\tau^*)^2$, and given $\theta, H, H^* \in \mathbb{R}$ satisfying $H^* + i\tau^* = e^{i\theta}(H + i\tau)$, there is a isometric correspondence between simply-connected H-surfaces immersed in $\mathbb{E}$ and simply-connected H^*- surfaces immersed in $\mathbb{E}^*$ (see [Dan07] for a more detailed description).

  By choosing $\kappa = \epsilon \in \{1, -1\}$, $\theta = \frac{\pi}{2}$ and restricting to the minimal case $H^* = 0$, we obtain $\tau = 0$ and, thus, a isometric correspondence between minimal immersions

$\phi : \Sigma \looparrowright \mathbb{E}(4H^2+\epsilon, H)$ and immersions $\phi^* : \Sigma \looparrowright \mathbb{M}^2(\epsilon)\times\mathbb{R}$ with mean curvature H, where Σ is simply-connected. The condition $\theta = \frac{\pi}{2}$ allows us to understand the behaviour of the corresponding surface when the original one contains a vertical or horizontal ambient geodesic (here, a geodesic is called vertical when it is tangent to the Killing field ξ, and horizontal when it is orthogonal to ξ). Unfortunately, the control of elements which are neither vertical nor horizontal seems to be difficult and we will only be able to consider polygons made out of vertical and horizontal geodesics, which is a quite rigid condition in $\mathbb{E}(4H^2+\epsilon, H)$.

In order to illustrate the correspondence, we will apply it to show that the spherical helicoids in the Berger spheres (i.e., minimal surfaces ruled by horizontal geodesics and invariant under a 1-parameter group of helicoidal motions) correspond to rotationally invariant H-surfaces in $\mathbb{M}^2(\epsilon)\times\mathbb{R}$ plus an example of different nature for $\epsilon = -1$; namely, the H-cylinder invariant under hyperbolic translations.

- The second step is to guarantee the existence of graphic solutions of the Plateau problem with boundary a certain geodesic polygon in $\mathbb{E}(\kappa,\tau)$. This can be achieved for a wide class of possible boundaries, known as Nitsche graphs. Roughly speaking, a Nitsche graph is a piecewise regular curve $\Gamma \subset \mathbb{E}(\kappa,\tau)$ satisfying: (a) it maps injectively by the standard projection $\pi : \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ onto $\partial\Omega$, Ω being an simply-connected open set in $\mathbb{M}^2(\kappa)$, except for some vertical segments contained in Γ, and (b) $\pi^{-1}(\overline{\Omega})$ is a mean-convex 3-manifold with boundary in which Γ is null-homotopic. We will prove that, if Γ is a Nitsche graph, then there exists a unique minimal surface $\Sigma \subset \pi^{-1}(\overline{\Omega})$ with boundary Γ and the interior of Σ is a vertical graph over Ω.

The first application of these results which we will give is the construction of a 1-parameter family of complete singly-periodic H-surfaces in the product spaces $\mathbb{M}^2(\epsilon)\times\mathbb{R}$ for $4H^2+\epsilon > 0$, $\epsilon \in \{-1, 1\}$, which come from minimal surfaces in the Berger spheres $\mathbb{E}(4H^2+\epsilon, H)$. The resulting surfaces in $\mathbb{M}^2(\epsilon)\times\mathbb{R}$ are invariant by a discrete 1-parameter group of ambient isometries, consisting of rotations in $\mathbb{S}^2\times\mathbb{R}$ or hyperbolic translations in $\mathbb{H}^2\times\mathbb{R}$. A quite precise study of the original minimal piece in the Berger sphere will show that the final surface must have the shape of a *horizontal unduloid*.

We remark that the constructed sufaces are all symmetric with respect to a horizontal slice (though not neccesarily embedded) and have bounded height. Furthermore, their maximum heights vary continuously from the maximum height of a rotationally invariant H-sphere in $\mathbb{M}^2(\epsilon)\times\mathbb{R}$ and the maximum heigth of the rotationally invariant H-torus (for $\epsilon = 1$) or the H-cylinder invariant under hyperbolic translations (for $\epsilon = -1$). In fact, the construction gives a continuous deformation between these two examples. For $\epsilon = 1$ and for all $H > 0$, we will show that the construction provides compact H-surfaces.

The second application is the construction $1/2$-surfaces in $\mathbb{H}^2\times\mathbb{R}$ which have the symmetries of a tessellation of $\mathbb{H}^2$ by regular polygons. This is the horizontal analog in $\mathbb{H}^2\times\mathbb{R}$ of doubly-periodic minimal surfaces in $\mathbb{R}^3$. Furthermore, we will give some applications to the construction of compact $1/2$-surfaces in $M\times\mathbb{R}$, where M is a compact surface with constant Gaussian curvature -1 that can be realized by some gluing patterns in a regular tessellation. In particular, all topological surfaces with negative Euler characteristic can be produced with this method.

References

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