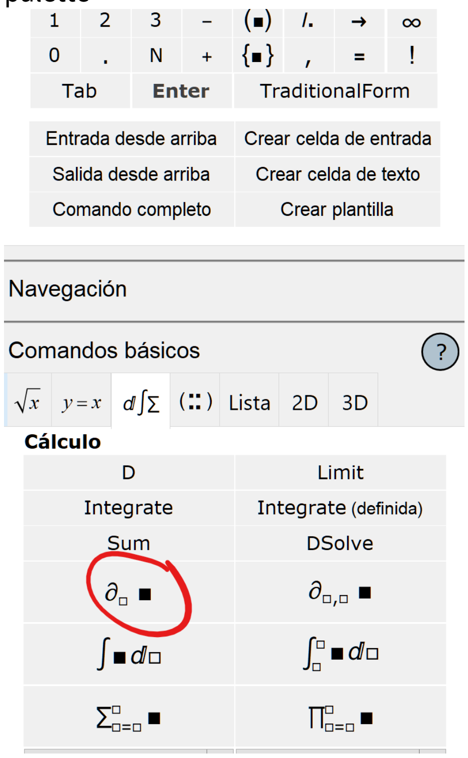
Exercise 2

```
Compute the Gauss curvature for X(u,v) =
 {-Cos[u] (2 + Cos[v]) + 2 Sin[v], (2 + Cos[v]) Sin[u] + 2 Sin[v], -Sin[v]}
  at the point (u,v) = (5,1).
```

To define a surface we use the usual notation for vectorial multivariate functions with Mathematica

In[3]:=
```
X[u_, v_] := {-Cos[u] (2 + Cos[v]) + 2 Sin[v], (2 + Cos[v]) Sin[u] + 2 Sin[v], -Sin[v]}
```

We can compute the tangen vector with the partial differential operator that we can find in the palette



Therefore the tangent vector $X_u(u, v)$ and $X_v(u, v)$ can be computed as

In[4]:=
```
∂u X[u, v]
```

Out[4]=
```
{(2 + Cos[v]) Sin[u], Cos[u] (2 + Cos[v]), 0}
```

In[5]:=
```
∂v X[u, v]
```

Out[5]=
```
{2 Cos[v] + Cos[u] Sin[v], 2 Cos[v] - Sin[u] Sin[v], -Cos[v]}
```

To compute the unit normal vector, it is convenient to define our own version of the Norm command of Mathematica to compute the norm of a vector in the following way

In[6]:=
```
norm[u_] := √(u.u)
```

The unit tangent vector is $N(u, v) = \frac{X_u(u,v)\times X_v(u,v)}{||X_u(u,v)\times X_v(u,v)||}$, and thus (we use samll letter n since capital letter N is already used by Mathematica for the numeric approximation command)

In[8]:=
```
n[u_, v_] := (∂u X[u, v] × ∂v X[u, v]) / norm[∂u X[u, v] × ∂v X[u, v]]
```

The partial derivatives of the unit normal vector $N_u(u, v)$ and $N_v(u, v)$ can be obtained again by means of the palette by means of $\partial_{\square}\,\blacksquare$. We know that $N_u(u, v)$ and $N_v(u, v)$ fall inside the tangent

plane and therefore they can be obtained as linear combinations of the tangent vectors $X_u(u, v)$ and $X_v(u, v)$ as
$N_u(u, v) = \alpha_u X_u(u, v) + \beta_u X_v(u, v)$
$N_v(u, v) = \alpha_v X_u(u, v) + \beta_v X_v(u, v)$

We can write this linear system with Mathematica as

```
In[12]:= {∂u n[u, v] == αu ∂u X[u, v] + βu ∂v X[u, v], ∂v n[u, v] == αv ∂u X[u, v] + βv ∂v X[u, v]}
```

We have to solve the corresponding linear system to compute $\alpha_u,\ \beta_u,\ \alpha_v,\ \beta_v$. With Mathematica we can solve such a system by means of the instruction Solve. We have to solve the mentiones variables $\alpha_u,\ \beta_u,\ \alpha_v,\ \beta_v$ so we have to indicate it to the instrucction. We use [[1]] to indicate that we want to extract the first and only solution from the result of Solve and we can also apply Simplify to obtain a more compact result.

```
In[16]:= solutions =
  Solve[{∂u n[u, v] == αu ∂u X[u, v] + βu ∂v X[u, v], ∂v n[u, v] == αv ∂u X[u, v] + βv ∂v X[u, v]},
     {αu, βu, αv, βv}][[1]] // Simplify

Out[16]= {αu → (Cos[v] (2 + Cos[v])^2 (3 + 4 Cos[v]^2 + 2 Cos[2 v] + 2 (Cos[u] - Sin[u]) Sin[2 v])) /
    (- (2 + Cos[v])^2
      (-3 + Cos[v]^2 (-2 + 8 Cos[u] Sin[u]) + 2 Sin[v]^2 - 2 (Cos[u] - Sin[u]) Sin[2 v]))^(3/2),
  βu → -((2 Cos[v]^2 (2 + Cos[v])^3 (Cos[u] + Sin[u])) / (- (2 + Cos[v])^2 (-3 +
          Cos[v]^2 (-2 + 8 Cos[u] Sin[u]) + 2 Sin[v]^2 - 2 (Cos[u] - Sin[u]) Sin[2 v]))^(3/2)),
  αv → -((2 Cos[v] (2 + Cos[v])^2 (1 + Cot[u]) Sin[u]) / (- (2 + Cos[v])^2 (-3 +
          Cos[v]^2 (-2 + 8 Cos[u] Sin[u]) + 2 Sin[v]^2 - 2 (Cos[u] - Sin[u]) Sin[2 v]))^(3/2)),
  βv → (2 + Cos[v])^3 / (- (2 + Cos[v])^2 (-3 + Cos[v]^2 (-2 + 8 Cos[u] Sin[u]) +
        2 Sin[v]^2 - 2 (Cos[u] - Sin[u]) Sin[2 v]))^(3/2)}
```

As the values obtained are quite difficult to retype, we stored the result in a variable called solutions.

Now we have to compute the determinant $\det\left(\begin{pmatrix} \alpha_u & \beta_u \\ \alpha_v & \beta_v \end{pmatrix}\right)$ and for this purpose we have to substitute in the determinant the values of $\alpha_u,\ \beta_u,\ \alpha_v,\ \beta_v$ obtained by means of Solve. To perform such a substitution we can use the substitution operator /.

```
In[20]:= Gausscurvature = Simplify[Det[( αu  βu
                                         αv  βv )] /. solutions]

Out[20]= Cos[v] / ((2 + Cos[v])
    (-3 + Cos[v]^2 (-2 + 8 Cos[u] Sin[u]) + 2 Sin[v]^2 - 2 Cos[u] Sin[2 v] + 2 Sin[u] Sin[2 v])^2)
```

We applied Simplify and stored the result in the variable Gausscurvature

Now we have to compute for the point (u,v)=(5,1) as it was indicated in the exercise. We substitute the values for u and v by means of a new substitution rule and we apply N to obtain a numeric result

```
In[22]:= N[Gausscurvature /. {u → 5, v → 1}]

Out[22]= 0.00829818
```

In case that the exercise asks the mean curvature, instead of the determinant, we have to compute

$\frac{1}{2}\,\text{trace}\left(\begin{pmatrix} \alpha_u & \beta_u \\ \alpha_v & \beta_v \end{pmatrix}\right)$. Therefore

In[24]:=
```
Meancurvature = Simplify[1/2 Tr[( αu βu
                                  αv βv )] /. solutions]
```

Out[24]=
$$-\left(\left((2+\text{Cos}[v])^2\left(-1-3\,\text{Cos}[v]^3-2\,\text{Cos}[v]^2\,(\text{Cos}[u]-\text{Sin}[u])\,\text{Sin}[v]+\text{Cos}[v]\,\left(-2+\text{Sin}[v]^2\right)\right)\right)\Big/\left(-(2+\text{Cos}[v])^2\left(-3+\text{Cos}[v]^2\,(-2+8\,\text{Cos}[u]\,\text{Sin}[u])+2\,\text{Sin}[v]^2-2\,\text{Cos}[u]\,\text{Sin}[2\,v]+2\,\text{Sin}[u]\,\text{Sin}[2\,v]\right)\right)^{3/2}\right)$$

And in the same way we should compute for the point (u,v)=(5,1)

In[25]:=
```
N[Meancurvature /. {u → 5, v → 1}]
```

Out[25]=
```
0.0961275
```