Grado en Ingeniería
Further Mathematics. List of exercises.
Chapter 3: Curves and surfaces

1) Consider the parameterized surface given by

$$\begin{array}{c} X : \mathbb{R}^2 \to \mathbb{R}^3 \\ X(u,v) = (u, v, u^2, v^2) \end{array}.$$

Determine the point wht the biggest Gauss curvature.

2) Consider the parameterized surface given by

$$\begin{array}{c} X : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \\ X(u,v) = (v^2 \cos(u), v^2 \sin(u), v) \end{array}.$$

Determine the point for which the Gauss curvature is negative.

3) Consider the parameterized surface given by

$$\begin{array}{c} X : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \\ X(u,v) = (v \cos(u), v \sin(u), v) \end{array}.$$

Compute the Gauss and mean curvature.

Remark: To compute the Gauss curvature we need first to obtain the unit norm vector (shape operator) given by

$$n(u,v) = \frac{X_u(u,v) \times X_v(u,v)}{\|X_u(u,v) \times X_v(u,v)\|}$$

Then we know that the partial derivatives of $n(u,v)$ lye on the tangent plane at the corresponding point $X(u,v)$ that is generated by mean of the two tangent vectors $X_u(u,v)$, $X_v(u,v)$. Therefore:

$$\begin{aligned} n_u(u,v) &\overset{\text{not}}{=} \frac{\partial n}{\partial u}(u,v) = \alpha_u X_u(u,v) + \beta_u X_v(u,v) \\ n_v(u,v) &\overset{\text{not}}{=} \frac{\partial n}{\partial v}(u,v) = \alpha_v X_u(u,v) + \beta_v X_v(u,v) \end{aligned}$$

Then the Weingarten transformation is

$$dn(u,v) = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}$$

and then we have

- The Gauss curvature at $X(u,v)$ is $K(u,v) = \det(dn(u,v)) = \alpha_u\beta_v - \alpha_v\beta_u$,
- The mean curvature at $X(u,v)$ is $H(u,v) = \frac{1}{2}\text{trace}(dn(u,v)) = \frac{1}{2}\left(\alpha_u + \beta_u\right)$.